Wolfgang Nolting

# Theoretical Physics 4

## Special Theory of Relativity

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Wolfgang Nolting
Inst. Physik
Humboldt-Universität zu Berlin
Berlin, Germany

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# General Preface

The nine volumes of the series *Basic Course: Theoretical Physics* are thought to be textbook materials for the study of university-level physics. They are aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. The conceptual design of the presentation is organized in such a way that

*Classical Mechanics (Vol. 1)*
*Analytical Mechanics (Vol. 2)*
*Electrodynamics (Vol. 3)*
*Special Theory of Relativity (Vol. 4)*
*Thermodynamics (Vol. 5)*

are considered as the theory part of an *integrated course* of experimental and theoretical physics as is being offered at many universities starting from the first semester. Therefore, the presentation is consciously chosen to be very elaborate and self-contained, sometimes surely at the cost of certain elegance, so that the course is suitable even for self-study, at first without any need of secondary literature. At any stage, no material is used which has not been dealt with earlier in the text. This holds in particular for the mathematical tools, which have been comprehensively developed starting from the school level, of course more or less in the form of recipes, such that right from the beginning of the study, one can solve problems in theoretical physics. The mathematical insertions are always then plugged in when they become indispensable to proceed further in the program of theoretical physics. It goes without saying that in such a context, not all the mathematical statements can be proved and derived with absolute rigor. Instead, sometimes a reference must be made to an appropriate course in mathematics or to an advanced textbook in mathematics. Nevertheless, I have tried for a reasonably balanced representation so that the mathematical tools are not only applicable but also appear at least "plausible."

The mathematical interludes are of course necessary only in the first volumes of this series, which incorporate more or less the material of a bachelor program. In the second part of the series which comprises the modern aspects of theoretical physics,

*Quantum Mechanics: Basics (Vol. 6)*
*Quantum Mechanics: Methods and Applications (Vol. 7)*
*Statistical Physics (Vol. 8)*
*Many-Body Theory (Vol. 9)*

mathematical insertions are no longer necessary. This is partly because, by the time one comes to this stage, the obligatory mathematics courses one has to take in order to study physics would have provided the required tools. The fact that training in theory has already started in the first semester itself permits inclusion of parts of quantum mechanics and statistical physics in the bachelor program itself. It is clear that the content of the last three volumes cannot be part of an *integrated course* but rather the subject matter of pure theory lectures. This holds in particular for *many-body theory* which is offered, sometimes, under different names, e.g., *advanced quantum mechanics*, in the eighth or so semester of study. In this part, new methods and concepts beyond basic studies are introduced and discussed which are developed in particular for correlated many-particle systems which in the meantime have become indispensable for a student pursuing master's or a higher degree and for being able to read current research literature.

In all the volumes of the series *Theoretical Physics*, numerous exercises are included to deepen the understanding and to help correctly apply the abstractly acquired knowledge. It is obligatory for a student to attempt on his own to adapt and apply the abstract concepts of theoretical physics to solve realistic problems. Detailed solutions to the exercises are given at the end of each volume. The idea is to help a student to overcome any difficulty at a particular step of the solution or to check one's own effort. Importantly, these solutions should not seduce the student to follow the *easy way out* as a substitute for his own effort. At the end of each bigger chapter, I have added self-examination questions which shall serve as a self-test and may be useful while preparing for examinations.

I should not forget to thank all the people who have contributed one way or another to the success of the book series. The single volumes arose mainly from lectures which I gave at the universities of Münster, Würzburg, Osnabrück, and Berlin (Germany), Valladolid (Spain), and Warangal (India). The interest and constructive criticism of the students provided me the decisive motivation for preparing the rather extensive manuscripts. After the publication of the German version, I received a lot of suggestions from numerous colleagues for improvement, and this helped to further develop and enhance the concept and the performance of the series. In particular, I appreciate very much the support from Prof. Dr. A. Ramakanth, a long-standing scientific partner and friend, who helped me in many respects, e.g., what concerns the checking of the translation of the German text into the present English version.

Special thanks are due to the Springer company, in particular to Dr. Th. Schneider and his team. I remember many useful motivations and stimulations. I have the feeling that my books are well taken care of.

Berlin, Germany Wolfgang Nolting
May 2016

# Preface to Volume 4

The main goal of Vol. 4 (*Special Theory of Relativity*) remains exactly the same as that of the total course on *theoretical physics*. It is thought to be an accompanying textbook material for the study of university-level physics. It aims to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. It is presented in such a way that it enables self-study without the need for a demanding and laborious reference to secondary literature. For the understanding of the text, it is only presumed that the reader has a good grasp of what has been elaborated in the preceding Vols. 1–3. Mathematical interludes are always presented in a compact and functional form and are practiced when they appear indispensable for the further development of the theory. Such mathematical insertions, though, are of course becoming decreasingly necessary with increasing volume number. For the whole text, it is true that I had to focus on the essentials, presenting them in a detailed and elaborate form, sometimes consciously sacrificing certain elegance. It goes without saying that after the basic course, secondary literature is needed to deepen the understanding of physics and mathematics.

The *special theory of relativity* belongs to the classical theories and is thus advisably taught immediately after *classical mechanics* (Vols. 1 and 2) and *electrodynamics* (Vol. 3). Due to this reason, the *special theory of relativity*, with its relativistic extension of *classical mechanics* and especially of *electrodynamics*, is presented as Vol. 4 in the underlying textbook series. The mathematically demanding nature of *electrodynamics* and therewith also of *special theory of relativity* makes practicing the application of concepts and methods especially mandatory. In this context, the exercises which are offered to each of the subsections play an indispensable role for effective learning. The elaborate solutions of exercises at the end of the book should not keep the learner from attempting an independent treatment of the problems but should only serve as a checkup of one's own efforts.

The *special theory of relativity* discussed in Vol. 4 deals with the dependence of physical statements on the reference system of the observer. Important in this connection are the inertial systems for which Newton's law of inertia is valid without any contribution of pseudo forces. According to *Einstein's principle of equivalence*, inertial systems basically are all on an equal footing. However, they are no longer transformed into each other by the *Galilean transformation* known from nonrelativistic mechanics but rather by the yet to be derived *Lorentz transformation*. The most fundamental consequence of Lorentz transformation consists in an intimate entanglement of space and time coordinates, from which a series of spectacular phenomena evolve. Some of them seem to even contradict the so-called common sense. Terms such as *space*, *time*, *simultaneity*, *etc*., must be thoroughly reconsidered. Einstein's second postulate states that the velocity of light in the vacuum at all space points once and for all is constant and in particular is completely independent of the kind of motion performed by the light source. From this postulate, the special form of the decisive Lorentz transformation matrix can be derived. The key issue of the *special theory of relativity* consists in verifying the physical laws and the cogent conclusions of *mechanics* and *electrodynamics* regarding their compatibility with respect to the Lorentz transformation between inertial systems. Deviations of the relativistically correct *mechanics* from the "familiar" *Newtonian mechanics* become evident above all when the relative velocities of physical systems are comparable to the velocity of light. The *special theory of relativity* leads, therefore, in this sense, to something like a *superordinate mechanics* that contains the nonrelativistic formulation as the limiting case for small relative velocities.

This volume on the *special theory of relativity* arose from lectures I gave at the German universities in Münster and Berlin. The animating interest of the students in my lecture notes has induced me to prepare the text with special care. The present one as well as the other volumes is thought to be the textbook material for the study of basic physics, primarily intended for students rather than for teachers.

I am thankful to the Springer company, especially to Dr. Th. Schneider, for accepting and supporting the concept of my proposal. The collaboration was always delightful and very professional. A decisive contribution to the book was provided by Prof. Dr. A. Ramakanth from the Kakatiya University of Warangal (India). He deserves a lot of thanks!

Berlin, Germany Wolfgang Nolting
May 2016

# Contents

# Chapter 1
# Basic Physical Principles

We start with a definition. Which concept is connected to the term

**theory of relativity?**

It is about the theory of the dependence or the invariance, as the case may be, of physical statements on the reference system of the observer. In particular, the

**special theory of relativity**

deals with the equal status of all **inertial systems**, where the transformations between the different inertial systems are no longer brought about by Galilean, but by

**Lorentz transformations.**

As we will see, this means an intimate and at first glance surprising entanglement of space and time coordinates. As the decisive starting points for the theory we will get to know two postulates, namely the

**equivalence principle**

and the

**principle of the constancy of velocity of light.**

The main consequence will lead to a revision of the terms

**space, time and simultaneity**

and will affirm the

**light velocity as the absolute limiting velocity**

and demonstrate the

**equivalence of energy and mass.**

 
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The Lorentz transformation refers only to uniformly straight-line relative motions of the considered systems and thus does not tell anything about systems which are accelerated relative to each other.

The

**general theory of relativity**

can be characterized as the theory of the fundamental equivalence of **all** space-time systems. Starting point here is the postulate of the proportionality between the heavy mass and the inertial mass (Sects. 2.2.1 and 2.2.2, Vol. 1). A very important result exposes the assumption that the space-time scheme has to be chosen euclidean as a prejudice. By a suitable choice of the metric a more transparent representation of the cosmos can be achieved. The space structure turns out to be dependent on the distribution of matter. The basic laws of mechanics follow in the framework of the *General Theory of Relativity* from the principle that a mass point, which is not influenced by electromagnetic forces, chooses in the space-time continuum just the *'shortest way'*. Problems such as the light-ray bending in the gravitational field of the sun or the red-shift of the spectral lines of atoms in strong gravitational fields find unique explanations by the general theory of relativity.

The mathematical procedure for finding the above-mentioned *'shortest way'* in a non-euclidean metric is not at all a simple task. However, the *General Theory of Relativity* is not the topic of this basic course. The reader must be referred to the relevant special literature.

Why and when is the *Special Theory of Relativity* necessary? The experimental experience teaches us that the postulates and definitions in the so far discussed form (Vol. 1) become invalid whenever the relative velocities $v$ come close to the velocity of light $c$:

$$v \approx c \quad (v \leq c) \; .$$

Then the *'relativistic corrections'* are indispensable, which are unimportant, though, for small velocities $v$. In this sense the theory of relativity represents in a way the completion of **classical** physics. From that it follows a *'new'* classical physics which correctly incorporates the *'old'* (Newtonian) physics (Vol. 1) as the limiting case for $v \ll c$.

Although quantum mechanics fulfills a similar functionality as a superordinate theory, a direct relationship between the theory of relativity and quantum mechanics does not yet exist. There are regions in which quantum effects turn out to be important while relativistic corrections are negligible and vice versa. The all in all superordinate *Relativistic Quantum Mechanics* deals with situations where both corrections are indispensable.

## 1.1 Inertial Systems

In the so-called *Newtonian mechanics*, which was reviewed in the first volume of this **'Basic Course: Theoretical Physics'**, fundamental terms such as the *trajectory* $\mathbf{r}(t)$ and the *velocity* $\mathbf{v} = \dot{\mathbf{r}}(t)$ of a mass point presuppose the existence of a reference system (coordinate system) as well as a time-measuring device ('clock'). For building coordinate systems, walls of a room, cardinal directions or something similar may serve, while as a clock mechanical systems with spring, balance wheel, and gear-wheels may be chosen as well as periodic motions like the earth's rotation, molecular oscillations etc.

The experimental observation now indicates that not in all reference systems the Newtonian mechanics works. For instance, in rotating coordinate systems it becomes correct only if certain rotation-caused *pseudo forces* (*inertia forces, centrifugal forces*) (Sects. 2.4 and 2.2.5, Vol. 1) are further added to the driving forces. That leads to the conception which is called

**Newton's fiction.**

and can be summarized by the following two points:

1. There does exist the **absolute space ('cosmic (world) ether')**. This is unchangeable and immovable and does not create any resistance towards the motion of material bodies. The motion of the *relative space* (partial space) relative to the absolute space may lead to the fact that the basic laws of mechanics are no longer valid. Only those relative spaces which are at rest or in a uniform straight-line motion within the absolute space leave the basic laws invariant.
2. It does exist an **absolute time**, i.e. a 'standard clock' existing somewhere in the cosmic ether.

Both postulates eventually prove to be untenable. Point 1 can at first be generalized in the sense that we do not postulate the *absolute space* but start from the indisputable fact that there indeed exist systems in which Newton's physics is undoubtedly valid. At this point we need to repeat some considerations of Sect. 2.2.3 in Vol. 1, in order to make it clear once more, in detail, which preconditions have been used.

**Definition 1.1.1** We denote as an **inertial system** a reference system in which Newton's law of motion (inertia)

$$\mathbf{F} = m\ddot{\mathbf{r}}$$

is valid without the assistance of any artificially introduced pseudo forces.

A system that rotates relative to an inertial system thus can **not** be an inertial system. We have learned in Sects. 2.2.4 and 2.2.5 of Vol 1 that then certain terms are to be added to the force equation which describe the rotation (centrifugal, Coriolis forces).

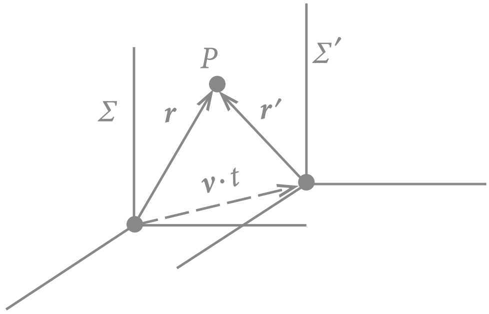


**Fig. 1.1** Two inertial systems moving uniformly in straight-line relative to each other. Demonstration of the Galilean transformation

**Assertion 1.1.1** Let Σ be an inertial system; if the system Σ′ moves uniformly in straight-line with respect to Σ and coincides with Σ at the time $t = 0$, then Σ′ is also an inertial system.

*Proof* Let $\mathbf{r}$ be the space vector for the point $P$ in Σ, $\mathbf{r}'$ that for $P$ in Σ′ (Fig. 1.1). Σ′ moves relative to Σ with the constant velocity $\mathbf{v}$. Then obviously:

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t \implies \dot{\mathbf{r}} = \dot{\mathbf{r}}' + \mathbf{v} \implies \ddot{\mathbf{r}} = \ddot{\mathbf{r}}' \; .$$

Hence:

$$\mathbf{F} = m\ddot{\mathbf{r}} = m\ddot{\mathbf{r}}' = \mathbf{F}' \quad \text{q. e. d.}$$

This proof uses as an important presumption that the time scale is the same in both the systems. For, when differentiating we have assumed as a matter of course that $t = t'$. Later we will see that this presumption must be seriously checked.

Without loss of generality we can assume that the constant relative velocity $\mathbf{v}$ lies parallel to the $z$-axis. Then the transition $\Sigma \longleftrightarrow \Sigma'$ is mediated by a

**Galilean transformation**

$$x = x' \; , \quad y = y' \; , \quad z = z' + vt \; , \quad t = t' \; . \tag{1.1}$$

The last relation is normally left out because of being apparently perfectly natural. In the case that the Galilean transformation is valid, we cannot detect by mechanical experiments a uniform straight-line movement (in rough approximation the path of the earth) relative to the *world ether*. However, perhaps this can be detected by optical experiments, e.g. by inspecting the velocity of light in different inertial systems:

A source of light at the origin of Σ emits spherical waves which propagate with the velocity of light $c$. For the space vector $\mathbf{r}$ of a certain point on the wave front it thus holds in Σ:

$$\dot{\mathbf{r}} = c\mathbf{e}_r \; ; \quad \mathbf{e}_r = \frac{\mathbf{r}}{r} \; .$$

But the wave velocity seen from $\Sigma'$ should then have the form, provided that the Galilean transformation is valid:

$$\dot{\mathbf{r}}' = c\mathbf{e}_r - \mathbf{v}$$

It means that the wave velocity would be dependent on the direction with $|\dot{\mathbf{r}}'| \neq c$. In $\Sigma'$ the waves would then be no longer spherical waves! In case this is true, then, we would have the possibility to define the *absolute space*. It would be just that reference system $\Sigma_0$ in which *spherical waves* are observed, i.e.:

$$\dot{\mathbf{r}} = c\mathbf{e}_r$$

All the other inertial systems would exhibit the above-mentioned directionality.

This can be checked rather simply experimentally!

## 1.2 Michelson-Morley Experiment

A.A. Michelson (Nobel prize 1907) designed an experimental set-up which should be able to measure with extreme accuracy the just discussed directionality of the wave velocity of light, provided it really exists. The arrangement is sketched in Fig. 1.2.

Coming from a light source $L$, a light beam hits a mirror $S_0$, which is covered on the front side by a metal layer so that it is semitransparent. A part of the beam is reflected at $S_0$, travels to the mirror $S_2$, is reflected there again, goes through $S_0$ and arrives finally at the observation device $B$ (telescope). The other part of the beam permeates $S_0$, impinges the mirror $S_1$, is there reflected and subsequently reflected once more at $S_0$, in order to interfere at the observation point $B$ with the first partial beam. Normally one puts into the course of beam $S_0 \rightleftarrows S_2$ a compensation plate

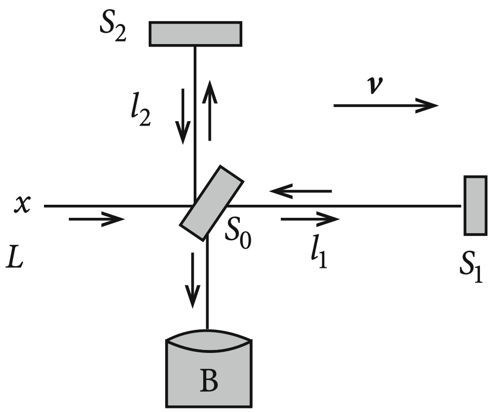


**Fig. 1.2** Schematic set-up of the Michelson-Morley experiment

in order to realize that this partial beam travels through the same thickness of glass as the other one. If monochromatic light is used then one observes at $B$ constructive interference of the two partial beams if their optical path lengths differ by an integer multiple of the wavelength $\lambda$:

$$\delta = (L_{02} + L_{20}) - (L_{01} + L_{10}) \stackrel{!}{=} m\,\lambda\ ; \quad m \in \mathbb{Z}\ .$$

$L_{ij}$ are the optical path lengths of the single sections:

$$L_{ij} = \int_{t_{S_i}}^{t_{S_j}} c\,dt = c\left(t_{S_j} - t_{S_i}\right); \qquad i,j = 0,1,2\ .$$

The transit times

$$\Delta_{ji} = t_{S_j} - t_{S_i}$$

are obviously dependent on the velocity of the ether if the absolute space does really exist and the light possesses in this space a non-directional velocity $c$. Let us now inspect in detail the **transit times** of the two partial beams:

1. Horizontal paths:

$S_0 \to S_1$: On this stretch we have to take into account the earth's velocity $\mathbf{v}$ which we assume to be along the horizontal. Assuming the validity of the Galilean transformation we can apply the additivity of the velocities. The velocity of the light relatively to the apparatus is therefore $c - v$. We obtain as transit time $\Delta_{10}$ for the path $S_0 \to S_1$:

$$\Delta_{10} = \frac{l_1}{c - v}\ .$$

$S_1 \to S_0$: On the way back the relative velocity amounts to $c + v$. The light is now travelling against the *ether wind*. That yields the transit time:

$$\Delta_{01} = \frac{l_1}{c + v}\ .$$

The total transit time of the first partial beam on the way $S_0 \to S_1 \to S_0$ is thus given by:

$$\Delta_1 = \frac{l_1}{c - v} + \frac{l_1}{c + v} = 2\frac{l_1}{c}\frac{1}{1 - v^2/c^2}\ . \tag{1.2}$$

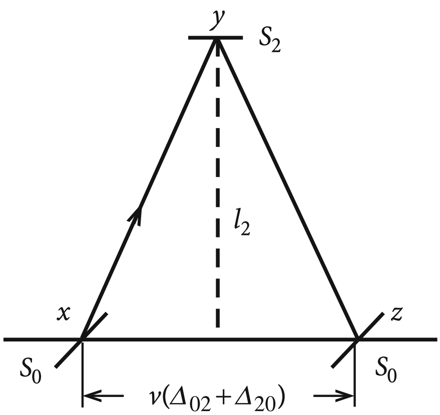


**Fig. 1.3** Calculation of the transit time of the light in the Michelson-Morley experiment

2. Vertical paths:

   The transit times for the forward and back path are now of course equal, $\Delta_{20} = \Delta_{02}$. However we have to take into account the co-movement of the mirror $S_0$ (Fig. 1.3). The velocity of the light is $c$ since the beam now always moves perpendicular to the ether wind. For the total way we then have:

$$\overline{xy} = c\,\Delta_{20} = \sqrt{l_2^2 + v^2\Delta_{20}^2} = \overline{yz}\,.$$

   This yields as the transit time $\Delta_2$ of the second partial beam:

$$\Delta_2 = \Delta_{20} + \Delta_{02} = \frac{2l_2}{c}\frac{1}{\sqrt{1 - v^2/c^2}}\,. \tag{1.3}$$

The two partial beams thus exhibit the following difference of the optical path lengths:

$$\delta = c(\Delta_2 - \Delta_1) = 2\left(\frac{l_2}{\sqrt{1 - v^2/c^2}} - \frac{l_1}{1 - v^2/c^2}\right) \overset{!}{=} m\lambda\,. \tag{1.4}$$

Now the apparatus is rotated by an angle of 90° so that the ways of the light $l_1$ and $l_2$ relatively to the ether are just interchanged. This leads to another path-length difference:

$$\delta' = c\left(\Delta_2' - \Delta_1'\right) = 2\left(\frac{l_2}{1 - v^2/c^2} - \frac{l_1}{\sqrt{1 - v^2/c^2}}\right) \overset{!}{=} m'\lambda\,. \tag{1.5}$$

Interesting now is the difference of the two path-length differences:

$$S = \delta' - \delta = 2\,(l_1 + l_2)\left(\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right)$$
$$= 2\,(l_1 + l_2)\left(1 + \frac{v^2}{c^2} + \ldots - 1 - \frac{1}{2}\,\frac{v^2}{c^2} + \ldots\right) .$$

This means:

$$S \underset{v^2 \ll c^2}{\longrightarrow} (l_1 + l_2)\,\frac{v^2}{c^2} \,. \tag{1.6}$$

$S$ causes a shift of the interference pattern by $r$ interference stripes where $r$ is given by

$$r = \frac{S}{\lambda} = \frac{(l_1 + l_2)\,v^2}{\lambda\,c^2} \,. \tag{1.7}$$

Hence, the rotation of the apparatus by $\pi/2$ in the described sense should lead to a shift of the interference stripes according to $S$. Let us estimate this by use of some realistic numerical values:

From the concept of the experiment it is of course not uniquely predetermined what is to be taken as *'ether velocity'* $v$. However, it is reasonable to accept the orbital velocity of the earth as $v$ so that

$$v = 3 \cdot 10^4\,\frac{\mathrm{m}}{\mathrm{s}}$$

is a good measure of the motion relative to the ether. Taking, furthermore, for the wavelength of the light $\lambda = 5000\,\text{Å} = 5 \cdot 10^{-7}\,\mathrm{m}$, one would observe a shift in the interference pattern by a full stripe, which means $S = \lambda$, provided

$$1 = (l_1 + l_2)\,\frac{9 \cdot 10^8}{9 \cdot 10^{16} \cdot 5 \cdot 10^{-7}\,\mathrm{m}} \Longleftrightarrow (l_1 + l_2) = 50\,\mathrm{m} \,.$$

Michelson performed his first experiment with a path length of

$$l_1 = l_2 = 1.2\,\mathrm{m}$$

This corresponds to a shift of about $0,05$ interference stripes and would have definitely been observable. In a subsequent experimental arrangement built up together with Morley the optical path length had been enhanced by a factor 10 by use of multiple reflections.

**Result:**

***Not the slightest*** *interference shift could be observed! The velocity of light is obviously the same in all directions and independent of the relative uniformly straight-line motions of the observer, the carrying medium and the source of the light.*

**Conclusion:**

*The Galilean transformation can not be correct! It has to be replaced by a transformation which guarantees the constancy of the velocity of light in* ***all*** *inertial systems.*

One should notice: The physical equivalence of inertial systems is not at all put into question, but only the type of transformation between such systems.

## 1.3 Einstein's Postulates

Einstein's interpretation of the unexpected outcome of the Michelson-Morley experiment was ingeniously simple. The problem concerning the interpretation of this experiment, in the last analysis, is a consequence of blind acceptance of the assumptions which appear to be plausible even though the relevant facts are not strictly proved. So it turns out that the assumption of an

**absolute time,**

being indispensable for the validity of the Galilean transformation, is as untenable as the assumption of an

**absolute space.**

How does one at all measure times? Each time measurement happens, strictly speaking, by an identification of a simultaneity. One compares, e.g., the hands of a clock with the arrival of a train. However, it might become problematic if one has to measure the time-correlation between two events which happen at different space points. In mechanics we try to describe the motion of a body by determining its position coordinates as functions of time. For this purpose we indeed need the time correlations between events at different space points. This is necessary even for any measurement of velocity since

$$\mathbf{v} = \frac{\mathbf{r}_a - \mathbf{r}_b}{t_a - t_b}$$

requires a time measurement $t_a$ at the position $\mathbf{r}_a$ and a time measurement $t_b$ at the position $\mathbf{r}_b$. But how is now the simultaneity at $\mathbf{r}_a$ related to that at $\mathbf{r}_b$? The $a$-clock and the $b$-clock have to be synchronized. That would not be a problem, though, if the information from $a$ to $b$ could be transmitted with infinitely high velocity. This is, however, impossible since even electromagnetic waves propagate with high but, in the end, finite velocity. However, the

**synchronization of clocks**

might be realized according to the following recipe. A light signal is emitted from $a$ to $b$ and then reflected by a mirror at $b$. The time needed for the distance $a \to b \to a$ is then measurable with the $a$-clock, only. This procedure makes sense, however, only under the very decisive precondition that the light velocity is exactly the same from $a$ to $b$ as from $b$ to $a$. For it holds then

$$t_{a\to b} = \frac{1}{2} t_{a\to b\to a} = t_{b\to a} \,,$$

and the synchronization of the $a$- and $b$-clocks would be possible without further ado.

Einstein has incorporated this precondition as postulate into his *new* physics which is called

**Special Theory of Relativity.**

The full theory is based on two postulates:

**Postulate 1.3.1 (Equivalence Principle)** *All physical laws and the results of all experiments are the same in all systems which move uniformly rectilinearly relative to one another.*

**Postulate 1.3.2 (Principle of the Constancy of the Velocity of Light)** *The velocity of light in the vacuum for all times and at all sites has the constant value c, and is in particular independent of the motion of its source.*

From Postulate 1.3.1 follows that only **relative** motions of two systems can be measured. Strictly speaking, this is nothing new compared to the Newtonian mechanics, only inertial systems are to be precisely defined. The actual new aspect is given by Postulate 1.3.2. Today it is uniquely experimentally confirmed; however, at the time, when Einstein proposed it, it was not at all commonly accepted, but rather, understandably, widely mistrusted. It provokes, as we will see, a radical rethinking with regard to familiar terms such as space, time, and simultaneity.

We are now left with the following issues:

1. We have to look for the correct transformation between inertial systems which keeps the velocity of light constant. It should go over into the Galilean transformation for $v \ll c$.
2. We have to check the physical laws concerning their transformation behavior with respect to such a correct transformation.

## 1.4 Lorentz Transformation

### *1.4.1 Matrix of Lorentz Transformation*

Let $\Sigma$ and $\Sigma'$ be inertial systems moving uniformly rectilinear to one another where we can assume, e.g., $\Sigma$ to be *at rest* and $\Sigma'$ to be *moving*. Both the systems shall be identical to each other at the time $t = 0$:

$$t = 0 : \qquad \Sigma \equiv \Sigma' \ .$$

At this point in time $t = 0$, a light source at the origin of $\Sigma$, which then just coincides with that of $\Sigma'$, emits a signal. This leads to a spherical wave propagating in the *rest system* $\Sigma$ with the velocity of light $c$:

$$c^2 t^2 = x^2 + y^2 + z^2 \ . \tag{1.8}$$

According to Postulate 1.3.2 this relation must hold for the light propagation in **all** inertial systems. i.e. must be fulfilled also in $\Sigma'$!

$$c^2 t'^2 = x'^2 + y'^2 + z'^2 \ . \tag{1.9}$$

The requirement that the signal in both systems is propagating as spherical wave can be satisfied obviously only by a co-transformation of the time ($t \Longleftrightarrow t'$). That results in an entanglement of space and time coordinates. But how does such a transformation, which transfers $\Sigma$ into $\Sigma'$, look like?

From reasons, which become clear later, we indicate the Cartesian coordinates from now on by superscript indexes:

$$\mathbf{r} = \left(x^1, x^2, x^3\right) = (x, y, z) \ .$$

It is common to introduce a fourth (or better 'zeroth') coordinate:

$$x^0 = c\,t \ . \tag{1.10}$$

The result of the Michelson-Morley experiment can be expressed as an

**invariance condition:**

$$\left(x^0\right)^2 - \sum_{\mu=1}^{3} (x^\mu)^2 \stackrel{!}{=} \left(x'^0\right)^2 - \sum_{\mu=1}^{3} \left(x'^\mu\right)^2 \tag{1.11}$$

We will see later (Sect. 1.5) that both the sides of this equation can be interpreted as the **length square** of a **four-vector (4-vector)** in an abstract four-dimensional

**Minkowski space.**

Then the required transformation, which we will already call now

**Lorentz transformation,**

obviously represents a

**rotation in the Minkowski space,**

where the *length* of the rotated four-vector does not change. To these rotations belong of course also the **normal** rotations of systems in the real three-dimensional visual space whose origins are at rest to one another. One can show:

*The **general Lorentz transformation** is equal to the special Lorentz transformation multiplied by the space rotation.*

By the

**special Lorentz transformation**

one understands the transformation between systems which move uniformly rectilinear relative to one another with parallel axes. The following considerations are restricted to these special Lorentz transformations. But then we can of course also assume that the relative velocity $\mathbf{v}$ between $\Sigma$ and $\Sigma'$ is directed parallel to the $x^3 = z$-axis.

The connection between the coordinates in $\Sigma$ and those in $\Sigma'$ must necessarily be linear because otherwise, e.g., a uniform rectilinear motion in $\Sigma$ would not be such a motion in $\Sigma'$, what would contradict the equivalence postulate. This explains the following **ansatz**:

$$x'^{\mu} = \sum_{\lambda=0}^{3} L_{\mu\lambda}\, x^{\lambda} \ . \tag{1.12}$$

Because of the special direction of $\mathbf{v}$ the 1- and 2-components will be the same in both systems:

$$x'^1 = x^1 \ ; \quad x'^2 = x^2 \ . \tag{1.13}$$

The components $x'^3$, $x'^0$ must be independent of $x^1$ and $x^2$ since no point of the $x^1$, $x^2$-plane is distinguished in any way. A shift in this plane must not have any impact. Therewith we already know the rough structure of the **transformation matrix L**:

$$\mathbf{L} \equiv \begin{pmatrix} L_{00} & 0 & 0 & L_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ L_{30} & 0 & 0 & L_{33} \end{pmatrix} \ . \tag{1.14}$$

We now exploit the invariance condition (1.11):

$$\left(x'^0\right)^2 - \left(x'^3\right)^2 \overset{!}{=} \left(x^0\right)^2 - \left(x^3\right)^2 \ .$$

This means:

$$\begin{aligned}\left(x'^0\right)^2 - \left(x'^3\right)^2 &= \left(L_{00}x^0 + L_{03}x^3\right)^2 - \left(L_{30}x^0 + L_{33}x^3\right)^2 \\ &= \left(L_{00}^2 - L_{30}^2\right)\left(x^0\right)^2 + \left(L_{03}^2 - L_{33}^2\right)\left(x^3\right)^2 \\ &\quad +2\left(L_{00}L_{03} - L_{30}L_{33}\right)x^0x^3 \ .\end{aligned}$$

Comparing coefficients yields:

$$\begin{aligned}L_{00}^2 - L_{30}^2 &= 1 \ , \\ L_{33}^2 - L_{03}^2 &= 1 \ , \\ L_{00}L_{03} - L_{30}L_{33} &= 0 \ .\end{aligned}$$

This system of equations can be solved by the following ansatz:

$$\begin{aligned}L_{33} = L_{00} &= \cosh\chi \ , \\ L_{30} = L_{03} &= -\sinh\chi \ .\end{aligned}$$

The arbitrariness of sign in the last row will later be eliminated by the requirement that for $v \ll c$ the Lorentz transformation must agree with the Galilean transformation! We therewith come to the intermediate result:

$$L \equiv \begin{pmatrix} \cosh\chi & 0 & 0 & -\sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix} \ .$$

To eventually also fix $\chi$, we consider the motion of the origin of $\Sigma'$. Seen from $\Sigma$ it holds for this special point:

$$x^3 = v\,t = \frac{v}{c}\,x^0 \ .$$

That yields the following relationship:

$$\begin{aligned}0 = x'^3 &= \cosh\chi\,x^3 - \sinh\chi\,x^0 = x^0\left(\frac{v}{c}\cosh\chi - \sinh\chi\right) \\ &\Longrightarrow \tanh\chi = \frac{v}{c} \ .\end{aligned}$$

With

$$\cosh\chi = \frac{1}{\sqrt{1-\tanh^2\chi}} = \frac{1}{\sqrt{1-v^2/c^2}}$$

and

$$\sinh\chi = \cosh\chi\ \tanh\chi = \frac{v/c}{\sqrt{1-v^2/c^2}}$$

as well as the usual abbreviations,

$$\beta = \frac{v}{c}\ ; \quad \gamma = \gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}\ , \tag{1.15}$$

we have finally found the

**matrix of the special Lorentz transformation**

$$L \equiv \begin{pmatrix} \gamma & 0\,0 & -\beta\gamma \\ 0 & 1\,0 & 0 \\ 0 & 0\,1 & 0 \\ -\beta\gamma & 0\,0 & \gamma \end{pmatrix} . \tag{1.16}$$

The determinant fulfills

$$\det L = \gamma^2 - \beta^2\gamma^2 = 1\ . \tag{1.17}$$

$L$ can therefore be interpreted as rotation (see Sect. 1.6.6, Vol. 1). The rows and columns of the matrix $L$, though, are not orthonormal. Because of their importance we want to write down once more the

**equations of the special Lorentz transformation**

in their original Cartesian form:

$$x' = x\ , \tag{1.18}$$

$$y' = y\ , \tag{1.19}$$

$$z' = \frac{z - v\,t}{\sqrt{1-v^2/c^2}} = \gamma\,(z - \beta c\,t)\ , \tag{1.20}$$

$$t' = \frac{t - \left(v/c^2\right) z}{\sqrt{1-v^2/c^2}} = \gamma\left(t - \frac{\beta}{c}\,z\right)\ . \tag{1.21}$$

We further add to this derivation, some discussion remarks:

1. For small relative velocities $v \ll c$ the Lorentz transformation (1.18) to (1.21) turns into the Galilean transformation (1.1).
2. $c$ is obviously the maximal relative velocity because for $v > c$ the coordinate $z'$ would not be real any more.
3. The inverse transformation matrix

$$L^{-1} \equiv \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \tag{1.22}$$

obviously results from (1.16) simply by the substitution $v \to -v$, i.e. $\beta \to -\beta$. That is of course absolutely to be expected since, seen from $\Sigma'$, the system $\Sigma$ moves with the velocity $-v$.

4. One denotes the *position vector* of the Minkowski space

$$x^{\mu} \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \tag{1.23}$$

as a *four-vector*. Each array of numbers

$$a^{\mu} \equiv \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \equiv \begin{pmatrix} a^0 \\ \mathbf{a} \end{pmatrix} , \tag{1.24}$$

which transforms by a Lorentz transformation in the same manner as the position vector (1.23) is likewise called a *four-vector*. In analogy to the position vector the 0-component is always denoted as *'time-component'* and the $(1, 2, 3)$-components as *'space-components'*. In particular, according to (1.11), the

$$\textbf{length square of} \;\; a^{\mu} : \left(a^0\right)^2 - \sum_{\mu=1}^{3} (a^{\mu})^2 = \left(a^0\right)^2 - \mathbf{a}^2$$

must be a **Lorentz invariant**. This definition will later become clear by the introduction of a corresponding scalar product. In Sect. 2.1 we will distinguish contravariant from covariant four-vectors and indicate that by different positions of the index $\mu$: $(a_{\mu},\, a^{\mu})$.

5. Let us agree upon some specialties of the notation. We index the components of four-vectors with Greek letters $\mu,\, \lambda,\, \rho, \ldots$ where the full four-vector is marked

by a typical component, e.g. $x^\mu$. The *'normal'* three-component vectors are represented by bold letters where Latin letters $k, l, m, \ldots$ denote their component indexes.

According to *Einstein's summation convention* one agrees to sum over the same Greek indexes of physical quantities standing next to each other where, for simplicity, the sigma-sign is frequently and intentionally omitted, e.g.:

$$x'^\mu = \sum_{\lambda=0}^{3} L_{\mu\lambda}\, x^\lambda \Longleftrightarrow x'^\mu = L_{\mu\lambda}\, x^\lambda \; . \tag{1.25}$$

Let us now discuss some important consequences of the Lorentz transformation.

### *1.4.2 Relativity of Simultaneity*

The consequence of the Lorentz transformation which goes most strongly against common sense concerns certainly the definition of *simultaneity*. Each inertial system has its own *criterion of simultaneity*. The absolute time does not exist! Time turns out to be a function of the used reference system and is fixed by the reading of a clock. The latter does not represent a problem at all as long as all time measurements are performed in one and the same inertial system. We could synchronize the clocks, as described previously, by a light signal. But how do we synchronize clocks which belong to different inertial systems which are on the move relative to one another? That turns out to be problematic since the term *simultaneity* is relative, i.e. different for different inertial systems $\Sigma$ and $\Sigma'$ which are moving relative to one another. That one can easily understand as follows:

$\Sigma$: Think that two *synchronized* clocks at $z_1$ and $z_2$ have determined that two events occur *simultaneously* at these sites:

$$t_1 = t_2 \iff \Delta t = 0 \; .$$

$\Sigma'$: The same two events, seen from this inertial system, though, do **not** appear as *coincidental*:

$$\begin{aligned} t_1' &= \frac{t_1 - (v/c^2) z_1}{\sqrt{1 - v^2/c^2}} \; ; \quad t_2' = \frac{t_1 - (v/c^2) z_2}{\sqrt{1 - v^2/c^2}} \\ &\Longrightarrow \Delta t' = t_1' - t_2' = \gamma \frac{v}{c^2} (z_2 - z_1) \neq 0 \; . \end{aligned} \tag{1.26}$$

It is now of course an interesting question whether even the sequence of two events may depend on the state of motion of the observer. Is it perhaps even possible to interchange cause and effect of a causal connection of two events? Let us clarify this point by the following consideration: In $\Sigma$ let $t_2 > t_1$. Then, the sequence of the events is surely retained in $\Sigma'$, too, provided:

$$0 < t_2' - t_1' = \gamma \left[ t_2 - t_1 - \frac{v}{c^2} (z_2 - z_1) \right] .$$

That means we must have

$$t_2 - t_1 > \frac{v}{c} \frac{z_2 - z_1}{c} .$$

Since $v < c$ the sequence remains definitely unchanged for

$$t_2 - t_1 \geq \frac{z_2 - z_1}{c} .$$

If the two events are **causally connected** in $\Sigma$ then the information process, which links cause and effect, occurs with **finite** velocity $\bar{v} \leq c$. This means:

$$t_2 - t_1 = \frac{z_2 - z_1}{\bar{v}} \geq \frac{z_2 - z_1}{c} .$$

Hence, cause and effect can **not** be interchanged. The sequence of not causally connected events, however, can indeed be reversed when observed from $\Sigma'$.

### *1.4.3 Time Dilatation*

In the inertial system $\Sigma$, a clock at the position $z$ emits two light signals with the time separation

$$\Delta t = t_1 - t_2 .$$

In the moving system $\Sigma'$ these light signals are observed at the times

$$t_1' = \gamma \left( t_1 - \frac{v\,z}{c^2} \right) ; \quad t_2' = \gamma \left( t_2 - \frac{v\,z}{c^2} \right) ,$$

i.e. with a separation:

$$\Delta t' = t_1' - t_2' = \gamma\, \Delta t = \frac{\Delta t}{\sqrt{1 - (v^2/c^2)}} > \Delta t . \tag{1.27}$$

The time interval $\Delta t$ appears extended for the moving observer in $\Sigma'$. He will say: *The stationary clock is slow!* The same will be stated, by the way, also by a person in $\Sigma$ who observes a clock in $\Sigma'$.

This phenomenon which appears paradoxial at first glance becomes less mysterious if one inspects the measuring process in detail. In $\Sigma$ (*at rest*) two events$(z, t_1)$ and $(z, t_2)$ are measured with **one and the same** clock at $z$. In contrast, in $\Sigma'$ (*moving*) the measurement needs **two** clocks, namely one at

$$z_1' = \gamma\,(z - v\,t_1)\ ,$$

and the other at

$$z_2' = \gamma\,(z - v\,t_2)\ ,$$

i.e. with the separation:

$$z_1' - z_2' = \gamma\, v(t_2 - t_1) \neq 0\ .$$

The measuring processes in the two inertial systems are thus not at all equivalent, hence, the results cannot be paradoxical, either. In $\Sigma'$ we have to synchronize the clocks which are placed at different sites. This synchronization leads, in the end, to the effect of the time dilatation. The time span, which is observed by one and the same clock at one and the same site, is called the

**proper time $\Delta\tau$.**

It is always smaller than the difference of the two timings in the *moving* system $\Sigma'$.

Today the phenomenon of time dilatation is observable by *almost everyday* experiments. One can exploit the radioactive decay of unstable particles for rather exact time measurements. The law of radioactive decay delivers a precise prediction how many of the at the time $t = 0$ present particles are not yet decayed at the time $t > 0$. The number of the not yet decayed particles is thus a measure of the elapsed time. This effect is used, for instance, for the age estimation of prehistorical excavations with the aid of unstable $C^{14}$-isotopes. According to B. Rossi and D.B. Hall (Phys. Rev. **59**, 223 (1941)) the time dilatation can experimentally be verified very impressively as follows:

1. $\mu$-mesons, which are created during the penetration of the cosmic radiation into the earth's atmosphere, are positively or negatively charged, and unstable.

$$\mu^\pm \longrightarrow e^\pm + \nu_1 + \overline{\nu}_2\ ,$$

| | |
|---|---|
| $e^\pm$ : | electron (positron) , |
| $\nu_1$ : | neutrino , |
| $\overline{\nu}_2$ : | antineutrino . |

2. $\mu^\pm$ hits the detector, comes there to rest, and decays after a certain time according to 1. Both events, the arrival of the $\mu^\pm$ in the detector as well as the emitting of the $e^\pm$, are detectable. The law of decay is therefore known.

3. Two detectors, one on a hill of height $L$, the other on sea level, count the incoming $\mu$-mesons per unit-time .
4. The velocity of the mesons is close to $c$:

$$v\left(\mu^{\pm}\right) \approx 0.994\, c\ .$$

The travel time $t_{tr}$ for the trip between the two detectors is therewith calculable. By the use of the law of decay it is then possible to determine the number of not yet decayed particles which are expected to reach the second detector.
5. Observation: Many more $\mu$-mesons than the expected number arrive at the second detector.
6. Explanation: The number of particles that actually arrive is not determined by $t_{tr}$, but by the proper time $\tau_{tr}$. The law of decay follows the co-moving clock:

$$\tau_{tr} = \frac{t_{tr}}{\gamma} \approx \frac{1}{9}\, t_{tr} \qquad (= 0.109\, t_{tr})\ .$$

The mesons which move with the velocity $v \approx 0.994\, c$ relative to the detectors represent a clock which is slow by a factor $1/9$.

### *1.4.4 Contraction of Length*

How is a **length measurement** to be performed? One places a scale on the length to be measured and reads out **simultaneously** the positions of the end points. That sounds trivial, if the length and the reference system $\Sigma$ are in relative rest to one another:

$$l = z_1 - z_2\ .$$

For the length measurement in the inertial system $\Sigma'$, which is moving relative to $\Sigma$ with the velocity $v$, we need at first the positions of the end points:

$$z_1' = \gamma\,(z_1 - v\,t_1)\ ; \quad z_2' = \gamma\,(z_2 - v\,t_2)\ .$$

What is to be inserted for $t_1,\ t_2$? The reading out has to be done simultaneously also in $\Sigma'$, i.e. it must be $t_1' = t_2'$, and not just $t_1 = t_2$. This means according to (1.21):

$$t_1 - \frac{v}{c^2} z_1 \stackrel{!}{=} t_2 - \frac{v}{c^2} z_2\ .$$

Hence it is

$$t_1 - t_2 = \frac{v}{c^2}\,(z_1 - z_2)$$

and therewith

$$l' = z_1' - z_2' = \gamma \left[ z_1 - z_2 - \frac{v^2}{c^2}(z_1 - z_2) \right] .$$

This means eventually:

$$l' = l\sqrt{1 - \frac{v^2}{c^2}} . \tag{1.28}$$

A rod of length $l$, at rest in $\Sigma$, appears in $\Sigma'$ contracted by the factor $(1-\beta^2)^{1/2} < 1$. The decisive point is that the length measurement prescribes that the positions of the end edges have to be read out **simultaneously**. The criterion of simultaneity, however, is different for different inertial systems. That influences the results of the length measurements.

## 1.4.5 Addition Theorem of Velocities

Could it not be possible to reach, by a sequence of Lorentz transformations, relative velocities which are even greater than the velocity of light $c$?

$$\underbrace{\Sigma_1 \underset{v_1}{\longrightarrow} \Sigma_2 \underset{v_2}{\longrightarrow} \Sigma_3}_{v_3 \longrightarrow} ; \qquad \mathbf{v}_i = v_i \mathbf{e}_z , \quad i = 1, 2, 3 .$$

If simply $v_3 = v_1 + v_2$ were to be taken, then, e.g., from $v_1 > c/2$ and $v_2 > c/2$ it would have to follow $v_3 > c$. But this would contradict Einstein's postulate.

Let us assume that the relative velocities $v_1, v_2, v_3$ are all oriented in $z$-direction:

$$\gamma_i = \left(1 - \beta_i^2\right)^{-(1/2)} ; \quad \beta_i = \frac{v_i}{c} ; \quad i = 1, 2, 3 . \tag{1.29}$$

Then it holds at first for the direct transition:

$\boxed{\Sigma_1 \rightarrow \Sigma_3 :}$

$$x^\mu_{(3)} = \widehat{L}_3 x^\mu_{(1)} ,$$

$$\widehat{L}_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix} . \tag{1.30}$$

Identical results must come out if we change from $\Sigma_1$ to $\Sigma_3$ via $\Sigma_2$:

$\boxed{\Sigma_1 \to \Sigma_2 \to \Sigma_3 :}$

$$x^\mu_{(3)} = \left(\widehat{L}_2 \widehat{L}_1\right) x^\mu_{(1)} ,$$

$$\widehat{L}_2 \widehat{L}_1 = \begin{pmatrix} \gamma_2 & 0 & 0 & -\beta_2\gamma_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_2\gamma_2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\beta_1\gamma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_1\gamma_1 & 0 & 0 & \gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 & -\gamma_1\gamma_2(\beta_1+\beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{pmatrix} . \tag{1.31}$$

The comparison of (1.30) and (1.31) leads to

$$\gamma_3 = \gamma_1\gamma_2 \left(1 + \beta_1\beta_2\right) ,$$
$$\beta_3\gamma_3 = \gamma_1\gamma_2 \left(\beta_1 + \beta_2\right) .$$

From that we get the *'addition theorem of relative velocities'*:

$$\beta_3 = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} . \tag{1.32}$$

It is therewith in any case $\beta_3 = (v_3/c) < 1$, if $\beta_1, \beta_2 < 1$. This can be read off directly from (1.32):

$$1 - \beta_3 = \frac{(1-\beta_1)(1-\beta_2)}{1 + \beta_1\beta_2} > 0 . \tag{1.33}$$

Therefore in all situations $c$ remains the limiting velocity! Let us discuss two **special cases**:

1. $v_1 = v_2 = 1/2\,c$:

   In this case it is $\beta_1 = \beta_2 = 1/2$ and therewith $\beta_3 = 4/5$:

$$v_3 = \frac{4}{5}c \neq v_1 + v_2 .$$

2. $v_1 = c\,;\ v_2 \le c$ arbitrary:
   It is now $\beta_1 = 1$, so that, according to (1.32), $\beta_3$ becomes independent of $v_2$:

$$\beta_3 = \frac{1+\beta_2}{1+\beta_2} = 1\ .$$

   This corresponds to Postulate 1.3.2 from Sect. 1.3. The light emitted from a source propagates in the vacuum with the velocity $c$, independently of the velocity $v$ of the light source.

Let us finally generalize the considerations of this chapter a bit. Let $\Sigma$ and $\Sigma'$ be two inertial systems for which the formulae (1.18) to (1.21) of the Lorentz transformation are valid. Let an object in $\Sigma$ have the velocity

$$\mathbf{u} \equiv \left(u_x,\ u_y,\ u_z\right) = \left(\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}\right)\ . \tag{1.34}$$

What is then its velocity in $\Sigma'$?

$$\mathbf{u}' \equiv \left(u'_x,\ u'_y,\ u'_z\right) = \left(\frac{dx'}{dt'},\ \frac{dy'}{dt'},\ \frac{dz'}{dt'}\right)\ . \tag{1.35}$$

The Lorentz transformation yields:

$$\begin{aligned}
dx' &= dx\ ,\\
dy' &= dy\ ,\\
dz' &= \gamma(dz - v\,dt)\ ,\\
dt' &= \gamma\left(dt - \frac{v}{c^2}\,dz\right) = \gamma\left(1 - \frac{v\,u_z}{c^2}\right)\,dt\ .
\end{aligned}$$

We obtain therewith the components of the velocity in $\Sigma'$:

$$u'_x = \frac{dx'}{dt'} = \frac{1}{\gamma}\,\frac{u_x}{1 - \frac{v\,u_z}{c^2}}\ , \tag{1.36}$$

$$u'_y = \frac{dy'}{dt'} = \frac{1}{\gamma}\,\frac{u_y}{1 - \frac{v\,u_z}{c^2}}\ , \tag{1.37}$$

$$u'_z = \frac{dz'}{dt'} = \frac{u_z - v}{1 - \frac{v\,u_z}{c^2}}\ . \tag{1.38}$$

Analogously one finds for an object in $\Sigma$ if it possesses the velocity $\mathbf{u}'$ in $\Sigma'$:

$$u_x = \frac{1}{\gamma}\,\frac{u'_x}{1+\dfrac{v\,u'_z}{c^2}}\,, \tag{1.39}$$

$$u_y = \frac{1}{\gamma}\,\frac{u'_y}{1+\dfrac{v\,u'_z}{c^2}}\,, \tag{1.40}$$

$$u_z = \frac{u'_z + v}{1+\dfrac{v\,u'_z}{c^2}}\,. \tag{1.41}$$

Let us check at the end the Lorentz invariance of the velocity of light, i.e. we verify whether with $\mathbf{u}^2 = c^2$ we also get $\mathbf{u}' = c^2$ as required from the special theory of relativity.

Let $\mathbf{u}^2 = c^2$:

$$\begin{aligned}
\mathbf{u}' &= \frac{1}{\gamma^2}\,\frac{1}{\left(1-\dfrac{v\,u_z}{c^2}\right)^2}\left(u_x^2+u_y^2\right) + \frac{(u_z - v)^2}{\left(1-\dfrac{v\,u_z}{c^2}\right)^2}\\
&= \left(1-\frac{v\,u_z}{c^2}\right)^{-2}\left[\left(1-\frac{v^2}{c^2}\right)\left(u_x^2+u_y^2\right) + u_z^2 + v^2 - 2v\,u_z\right]\\
&= \left(1-\frac{v\,u_z}{c^2}\right)^{-2}\left[c^2 - \frac{v^2}{c^2}\left(c^2-u_z^2\right) + v^2 - 2v\,u_z\right]\\
&= \frac{c^2}{\left(1-\dfrac{v\,u_z}{c^2}\right)^2}\left[1+\frac{v^2\,u_z^2}{c^4} - 2\frac{v\,u_z}{c^2}\right] = c^2\,; \quad \text{q.e.d.}
\end{aligned}$$

## 1.5 Light Cone, Minkowski Diagram

We go back once more to the general results of Sect. 1.4.1 and develop a sometimes quite useful geometrical illustration of the special theory of relativity.

With Eq. (1.23) we have already got to know the *position vector* of the Minkowski space:

$$x^\mu \equiv \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \equiv (c\,t, \mathbf{x})\,. \tag{1.42}$$

The *length square*

$$s^2 = c^2t^2 - \mathbf{x}^2 = c^2t^2 - \sum_{\mu=1}^{3} (x^\mu)^2 \tag{1.43}$$

is according to Postulate 1.3.2 of the special theory of relativity a **Lorentz invariant**, i.e. a physical quantity which does not change in consequence of a Lorentz transformation.

We can represent the position vector (1.42) in a space-time diagram, the so-called

**Minkowski diagram,**

whose axes are given by $x, y, z$ and $c\,t$. For the time axis one applies $c\,t$, in order that all axes have the dimension of a length. Since the $x$- and $y$-components remain invariant with the appropriate Lorentz transformations, we can fix $x = y = 0$.

Each point $P$ of the Minkowski space represents an **'event'**. Its coordinates are the axes intercepts which are found when one puts straight lines parallel to the axes through the point $P$. As **light signal** one denotes the straight line through the origin defined by $s^2 = 0$. In the case of equal scaling of the space- and time-axes it is just the angle bisector in the $z$-$ct$-diagram (Fig. 1.4).

The description of an event in the Minkowski diagram can be carried out in arbitrarily many different manners, corresponding to the respective reference system. The inertial system, in which the space- and time-axes are perpendicular to one another, is by no means physically special compared to $\Sigma'$ whose axial directions can be determined as follows: We assume that the origins of coordinates of $\Sigma$ and $\Sigma'$ coincide at the time $t = t' = 0$. The

**$\Sigma'$-time axis**

is then defined by $z' \equiv 0 \equiv \gamma(z - v\,t)$. This means $z = v\,t$ or

$$c\,t = \frac{1}{\beta} z\,. \tag{1.44}$$

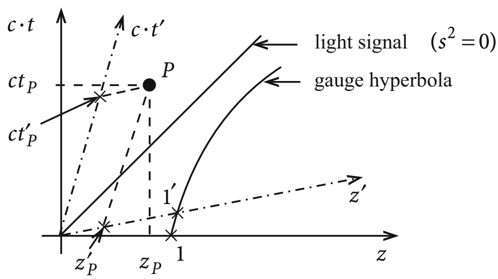


**Fig. 1.4** To the construction of a Minkowski diagram

The $\Sigma'$-time axis is thus in $\Sigma$ a straight line with the slope $(1/\beta) > 1$. It is therefore always located between the $\Sigma$-time axis and the light signal (Fig. 1.4). The

**$\Sigma'$-space axis**

is fixed by $t' \equiv 0 \equiv \gamma\left(t - (v/c^2)z\right)$. That means in this case:

$$c\,t = \beta\,z\,. \tag{1.45}$$

The space axis thus represents in $\Sigma$ a straight line with the slope $\beta < 1$, therefore lies always between the $\Sigma$-space axis and the light signal.

Of course, also the scaling of the axes will change after the Lorentz transformation $\Sigma \to \Sigma'$. The

**gauging of the axes**

is done according to the following recipe: Since $s^2$ is a Lorentz invariant and $x$ and $y$ are not affected by the transformation, the expression

$$\hat{s}^2 = s^2 + x^2 + y^2 = (c\,t)^2 - z^2$$

must also be a Lorentz invariant. The geometric locus of all points which obey

$$\hat{s}^2 = -1 \Longleftrightarrow z^2 = (c\,t)^2 + 1$$

represents in $\Sigma$ an equilateral hyperbola which intersects the $z$-axis ($t = 0$) at $z = 1$ (Fig. 1.4). The scale unit in $\Sigma$ is therewith fixed. All points of the hyperbola correspond to position vectors of the (projected) *length* $\hat{s}^2 = -1$. But since this is a Lorentz invariant, these position vectors all have in $\Sigma'$ also the (projected) *length* $-1$. They hence fulfill the relation

$$z' = \left(c\,t'\right)^2 + 1\,.$$

The intersection point the gauge hyperbola with the $z'$-axis fixes therewith the scale unit $z' = 1$ (see Fig. 1.4).

Analogously, the point of intersection of the hyperbola, defined by

$$\hat{s}^2 = +1 \Longleftrightarrow (c\,t)^2 = z^2 + 1\,,$$

with the $t$-axis ($z = 0$) provides the time unit in $\Sigma$, the intersection point with the $t'$-axis ($z' = 0$) the time unit in $\Sigma'$. The gauging of the axes is therewith complete.

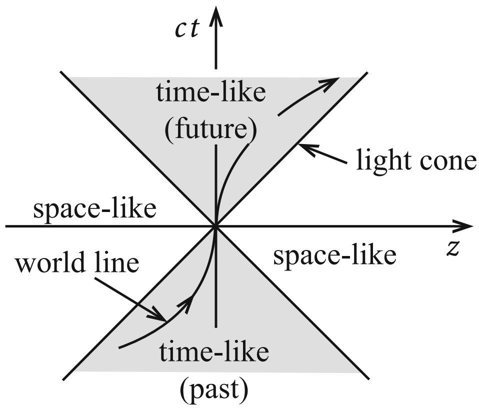


**Fig. 1.5** Light cone and world line in the Minkowski space

The *length square* of a four-vector is, as already exploited several times, not necessarily positive. One therefore distinguishes:

$$s^2 = (c\,t)^2 - \mathbf{x}^2 \begin{cases} > 0 : & \textit{time-like} \text{ four-vector} , \\ = 0 : & \textit{light-like} \text{ four-vector} , \\ < 0 : & \textit{space-like} \text{ four-vector} . \end{cases} \tag{1.46}$$

The Minkowski space can be decomposed accordingly (Fig. 1.5). All time-like four-vectors are found within the so-called

**light cone**

whose surface is defined by $s^2 = 0$. Because of $v \le c$ the trajectories of physical particles in the Minkowski space, which are called

**world lines** ,

lie, all of them, within the light cone provided they start at $t = 0$ at the origin. The world lines of the photons are located on the light cone. All **space-like** four-vectors are outside the light cone. Since $s^2$ is a Lorentz invariant, each four-vector retains in **all** inertial systems its character to be either space-like or time-like.

Let us finally consider in somewhat more detail the distance between two *world events* $P_1(c\,t_1, \mathbf{x}_1)$ and $P_2(c\,t_2, \mathbf{x}_2)$:

$$s_{12}^2 = c^2\,(t_1 - t_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2 \ . \tag{1.47}$$

With the four-vectors $x^\mu_{(1)} = (c\,t_1, \mathbf{x}_1), x^\mu_{(2)} = (c\,t_2, \mathbf{x}_2)$ the difference vector $x^\mu_{(1)} - x^\mu_{(2)}$ is of course also a four-vector and the length square $s_{12}^2$ (*space-time interval*) hence a Lorentz invariant. Without loss of generality for the following statements we can assume that $(\mathbf{x}_1 - \mathbf{x}_2)$ has the direction of the $z$-axis. It is therefore $|\mathbf{x}_1 - \mathbf{x}_2| = z_1 - z_2$ if $z_1 > z_2$.

1. **Space-like distance** ($s_{12}^2 < 0$)

   It follows from $s_{12}^2 < 0$ that $z_1 - z_2 > c\,(t_1 - t_2)$. This means that the two events $P_1$ and $P_2$ are not connectable by a light signal. Therefore,

   **no causal correlation**

   can exist between them. It is always possible to find a Lorentz transformation into an inertial system $\Sigma'$ where the two events $P_1$ and $P_2$ appear simultaneously:

   $$c\left(t_1' - t_2'\right) = \gamma\,(c\,(t_1 - t_2) - \beta\,(z_1 - z_2)) \stackrel{!}{=} 0\ .$$

   Because of $z_1 - z_2 > c(t_1 - t_2)$ it exists of course always a $\beta < 1$ with

   $$\beta\,(z_1 - z_2) \stackrel{!}{=} c\,(t_1 - t_2)\ ,$$

   so that $t_1' = t_2'$. The sequence of world events with space-like distances can always be interchanged by proper Lorentz transformations.
2. **Time-like distance** ($s_{12}^2 > 0$)

   $s_{12}^2 > 0$ means $c(t_1 - t_2) > z_1 - z_2$. The world events $P_1$ and $P_2$ are therefore bridgeable by a light signal. A

   **causal correlation is possible!**

   However, because of $c(t_1 - t_2) > z_1 - z_2$ and therewith even more

   $$c\,(t_1 - t_2) > \beta\,(z_1 - z_2)$$

   by **no** Lorentz transformation a simultaneity is reachable. Cause and effect can **not** be interchanged.

   Because of

   $$z_1' - z_2' = \gamma\,[(z_1 - z_2) - v\,(t_1 - t_2)]$$

   one can transform, though, into an inertial system for which $z_1' = z_2'$ so that the events take place at the same site.

   The special case $s_{12}^2 = 0$ is denoted as *light-like* distance.

## 1.6 Exercises

**Exercise 1.6.1** A spacecraft moves with the velocity $v = 0.8\,c$. As soon as it reaches a distance $d = 6.66 \cdot 10^8$ km from the earth, a radio signal is emitted from the earth station to the spaceship. How much time does the signal need

1. according to a clock at the earth station,
2. according to a clock in the spaceship.

**Exercise 1.6.2** Let $\Sigma$ and $\Sigma'$ be two inertial systems. $\Sigma'$ moves relative to $\Sigma$ with the velocity $v = (3/5)\,c$ in the $z$-direction. At the time $t = t' = 0$ both the systems coincide: $\Sigma = \Sigma'$. Let now an event in $\Sigma'$ have the coordinates:

$$x' = 10\,\text{m}\,; \quad y' = 15\,\text{m}\,; \quad z' = 20\,\text{m}\,; \quad t' = 4 \cdot 10^{-8}\,\text{s}\,.$$

Determine the coordinates of the event in $\Sigma$!

**Exercise 1.6.3** Let $\Sigma$ and $\Sigma'$ be two inertial systems. $\Sigma'$ moves relative to $\Sigma$ with the velocity $v$ in the $z$-direction. Two events happen in $\Sigma$ at the times $t_1 = z_0/c$ and $t_2 = z_0/2c$ at the sites $(x_1 = 0,\ y_1 = 0,\ z_1 = z_0)$ and $(x_2 = 0,\ y_2 = y_0, z_2 = 2z_0)$. How large must the relative velocity $v$ be to let the events happen simultaneously in $\Sigma'$? At which time $t'$ will the events then be observed in $\Sigma'$?

**Exercise 1.6.4** In an inertial system $\Sigma$ two events happen at the same place with a time separation of 4 s. Calculate the spatial distance of the two events in an inertial system $\Sigma'$, in which the events appear with a time separation of 5 s!

**Exercise 1.6.5** In an inertial system $\Sigma$ two simultaneous events have a spatial separation of 3 km on the $z$-axis. In $\Sigma'$ this distance amounts to 5 km. Calculate the constant velocity $v$, with which $\Sigma'$ moves relative to $\Sigma$ in $z$-direction. Which time separation do the events have in $\Sigma'$?

**Exercise 1.6.6** Let $\Sigma$ and $\Sigma'$ be two inertial systems. $\Sigma'$ moves relative to $\Sigma$ with the velocity $\mathbf{v}$, where the direction of $\mathbf{v}$ is arbitrary, i.e. not necessarily parallel to the $z$-axis of $\Sigma$. Find the explicit formulae of the Lorentz transformation! Derive the transformation matrix $\widehat{L}$! Write down $\widehat{L}$ for the special case $\mathbf{v} = v\,\mathbf{e}_x$!

**Exercise 1.6.7** Let $\Sigma$ and $\Sigma'$ be two inertial systems moving relative to one another with the velocity $\mathbf{v} = v\,\mathbf{e}_z = \text{const.}$

1. A rod at rest in $\Sigma$ is at angle of 45° to the $z$-axis. At which angle does it appear in $\Sigma'$?
2. A particle in $\Sigma$ has the velocity $\mathbf{u} = (v, 0, 2\,v)$. Which angle does its trajectory form with the $z$-axes in $\Sigma$ and $\Sigma'$?
3. A photon leaves the origin of $\Sigma$ at the time $t = 0$ in a direction which forms an angle of 45° with the $z$-axis. What is the angle in $\Sigma'$?

**Exercise 1.6.8** A rocket of the *'proper length'* $L_0$ flies with constant velocity $v$ relative to a reference system $\Sigma$ in $z$-direction. At the time $t = t' = 0$ the tip of the rocket passes the point $P_0$ in $\Sigma$. In this moment a light signal is sent out from the tip of the rocket to the end of the rocket.

1. After what time does the light signal reach the end of the rocket in the rest system of the rocket?
2. At which point of time does the signal reach the end of the rocket in the rest system $\Sigma$ of the observer?
3. When does the observer register that the end of the rocket passes the point $P_0$?

**Exercise 1.6.9** Let $\Sigma$, $\Sigma'$ be two inertial systems which are moving relatively to one another with the velocity $\mathbf{v} = v\,\mathbf{e}_z$. A particle in $\Sigma$ has the velocity

$$\mathbf{u} = \left(u_x, u_y, u_z\right) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) .$$

Let

$$\mathbf{u} = (0, c, 0) .$$

Calculate $\mathbf{u}'$!

**Exercise 1.6.10**

1. Can there be a causal connection between the following events?

$$\begin{aligned}
&\text{(a) } x_1 = 1\,\text{m}\,;\ y_1 = 2\,\text{m}\,;\ z_1 = 3\,\text{m}\,;\quad t_1 = 3\cdot 10^{-8}\,\text{s}\,,\\
&\phantom{\text{(a) }} x_2 = 4\,\text{m}\,;\ y_2 = 2\,\text{m}\,;\ z_2 = 7\,\text{m}\,;\quad t_2 = 6\cdot 10^{-8}\,\text{s}\,,\\
&\text{(b) } x_1 = 7\,\text{m}\,;\ y_1 = 0\,;\quad z_1 = -2\,\text{m}\,;\ t_1 = 1.1\cdot 10^{-7}\,\text{s}\,,\\
&\phantom{\text{(b) }} x_2 = 4\,\text{m}\,;\ y_2 = 5\,\text{m}\,;\ z_2 = +3\,\text{m}\,;\ t_2 = 0.9\cdot 10^{-7}\,\text{s}
\end{aligned}$$

2. Is it possible to find an inertial system in which these events appear simultaneously? With what a velocity and in which direction would this system have to move relative to that in part 1.?

**Exercise 1.6.11** $\mu$-mesons (muons) are generated when cosmic radiation penetrates the earth's atmosphere at the height

$$H \approx 3\cdot 10^4\,\text{m} .$$

In their rest system the $\mu$-mesons have a lifetime (proper time) of $\tau \approx 2\cdot 10^{-6}$ s. This means $c\tau \approx 600$ m. Nevertheless, almost all $\mu$-mesons reach the earth's surface. This one understands only if the velocity of the $\mu$-mesons is comparable with the velocity of light $c$.

1. How strong should the deviation of the muon velocity from $c$,

$$\varepsilon = \frac{c - v}{c} ,$$

   be in order that the muons reach the surface of the earth?
2. Which height of the particle generation $H'$ does an observer *feel*, who co-moves with the muon?

**Exercise 1.6.12** Light that comes from a source $L$ falls onto a semitransparent mirror $S_0$, and is partially reflected to $S_3$, and also partially transmitted to the mirror $S_1$. $S_1$ and $S_2$ are totally reflecting mirrors. In this way one finally gets two coherent partial beams which interfere at the telescope $B$. The light rays traverse on the whole the distance $2l$ in a tube, in which water is running with the flow velocity $v$, the

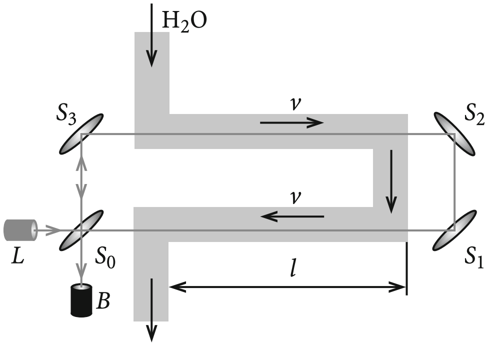


**Fig. 1.6** Schematic set up for the Fizeau-experiment (Exercise 1.6.12)

one partial ray parallel, the other antiparallel to the water flow (Fig. 1.6). In $B$ an interference pattern is observed which corresponds to the difference of the optical path lengths $c\,\Delta t$ with

$$\Delta t = 2l\left(\frac{1}{\frac{c}{n} - fv} - \frac{1}{\frac{c}{n} + fv}\right) .$$

$n$ is the index of refraction of the water. Calculate the *Fresnel's dragging coefficient* $f$ and demonstrate that it is compatible with Einstein's postulates and does not necessarily require Newton's fiction of the *world ether*. (Fizeau experiment.)

## 1.7 Self-Examination Questions

1. Which idea is behind the term *theory of relativity*?
2. What is understood by *Newton's fiction*?
3. What is an inertial system?
4. Define the Galilean transformation! What is said by this transformation about the times $t$ and $t'$ in the inertial systems $\Sigma$ and $\Sigma'$?
5. Describe the Michelson-Morley experiment!
6. What is the result of the Michelson-Morley experiment?
7. Formulate Einstein's postulates!
8. What does the matrix $\widehat{L}$ of the special Lorentz transformation say? Sketch its derivation!
9. Which connection exists between the times $t$ and $t'$ in inertial systems $\Sigma$ and $\Sigma'$ which are moving relative to one another in a uniform straight-line?
10. How can we find out directly from the transformation matrix that $c$ is the maximal relative velocity of inertial systems?
11. Which relationship exists between the Lorentz and Galilean transformations?
12. Interpret the relativity of simultaneity!

13. Is it possible to interchange cause and effect of a causal correlation simply by variation of the inertial system?
14. Describe the phenomenon of time dilatation!
15. What is denoted as *proper time*?
16. How can the time dilatation be experimentally proven?
17. How is a length measurement to be performed?
18. A rod resting in $\Sigma$ has the length $l$. What does a corresponding measurement of the length yield in the inertial system $\Sigma'$, which moves relative to $\Sigma$ with the velocity $v = \text{const}$?
19. What is the relativistic addition theorem for relative velocities?
20. Let $\Sigma_1, \Sigma_2, \Sigma_3$ be inertial systems. $\Sigma_2$ moves relative to $\Sigma_1$ in $z$-direction with the velocity $v_1 = c$, $\Sigma_3$ relative zu $\Sigma_2$ with $v_2 = c/2$. What is the velocity $v_3$ with which $\Sigma_3$ moves relative to $\Sigma_1$?
21. What does one understand by a Minkowski diagram?
22. What defines the light signal?
23. Let $\Sigma$ and $\Sigma'$ be two inertial systems which move relative to each other with the velocity $v = \text{const}$ in $z$-direction, while the origins of coordinates coincide at the time $t = t' = 0$. Space and time axes of $\Sigma$ are perpendicular to one another. How does one determine the space and time axes in $\Sigma'$?
24. How does one scale the axes of the Minkowski diagram?
25. What are time-like, light-like, space-like four-vectors?
26. Define the light cone!
27. Is it possible to convert by change of the inertial system a time-like into a space-like four-vector?
28. What is to be understood by a space-like (time-like) distance of two world events?
29. Why can a causal correlation between two world events with space-like distance not exist?
30. When can the sequence of two world events be interchanged by a proper Lorentz transformation, for space-like or for time-like distance?

# Chapter 2
# Covariant Four-Dimensional Representations

## 2.1 Covariant and Contravariant Tensors

### *2.1.1 Definitions*

In Sect. 1.4 we got to know the correct transformation between inertial systems which fulfills Postulate 1.3.2 of Sect. 1.3. The point now is to write down all the physical laws in

**covariant form,**

i.e. to formulate it in such a manner that it remains **form-invariant** under Lorentz transformations. That corresponds to the equivalence of all inertial systems according to Postulate 1.3.1.

The Newtonian laws of classical mechanics are form-invariant only under Galilean transformations which, as we now know, are correct only in the limit $v \ll c$. Consequently, in the relativistic region the basic laws of mechanics and also electrodynamics will no longer have the familiar form. Our next task must therefore be to check the form-invariance of the physical laws under Lorentz transformations. This inspection is most advisably done in the four-dimensional Minkowski space. The Lorentz transformation represents in this space a *rotation* of the four-vectors which keeps their *length squares* invariant.

Already in non-relativistic physics, form-invariance of physical laws under rotation in the normal three-dimensional space was to be required. This, however, was trivially fulfilled in general. A physical law is a mathematical equation. A

$$\textbf{scalar law}: \quad a = b$$

is of course invariant under rotations since neither $a$ nor $b$ changes thereby. A

$$\textbf{vectorial law}: \quad \mathbf{a} = \mathbf{b} \Longleftrightarrow a_j = b_j\,; \quad j = 1, 2, 3$$



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is covariant under rotations, i.e., the components change, but such that $a'_j = b'_j$ is valid for all $j$ and therewith $\mathbf{a}' = \mathbf{b}'$. Analogous statements hold for tensors of arbitrary rank.

The prescription is therewith clear: **Form-invariance** of a physical law under Lorentz transformations is realized if and only if the law is given in covariant four-dimensional form, i.e., if all the terms of the respective equation are

**four-tensors of the same rank**

Under this perspective we will reprocess in Sect. 2.2, the basic laws of mechanics and in Sect. 2.3 those of electrodynamics.

At first, however, we still have to comment in more detail on the above-mentioned *prescription*. For this purpose we present some formal considerations about calculations in the four-dimensional Minkowski space, where we have to in particular recall the tensor term introduced in Sect. 4.3.3 of Vol. 1. Actually it is nothing but an extension of the vector term. An $n^k$-tuple of numbers in an $n$-dimensional space represents a tensor of $k$-th rank if these numbers transform under a change of the system of coordinates ($\Sigma \to \Sigma'$) according to certain fixed rules. The interesting space here is the Minkowski space with $n = 4$. The change of coordinates is due to a Lorentz transformation which we have derived ultimately from the invariance of the *length square*

$$s^2 = \left(x^0\right)^2 - \mathbf{x}^2 = c^2t^2 - x^2 - y^2 - z^2$$

of the four-vector (1.42),

$$x^\mu \equiv (c\,t, \mathbf{x}) \; .$$

The transformation is linear

$$x'^\mu = L_{\mu\,\lambda}\, x^\lambda \; ,$$

where the matrix elements $L_{\mu\,\lambda}$ are defined by (1.16). Take notice of the summation convention (1.25). A

**tensor of $k$-th rank**,

which belongs to the space-time point $x^\mu$, is now defined by its transformation behavior under the transformation $x^\mu \to x'^\mu$. As to the Minkowski space, this tensor is a $4^k$-tuple of numbers which behave under the transformation of coordinates

$$x^\mu \to x'^\mu = L_{\mu\,\lambda}\, x^\lambda$$

according to certain fixed rules. The numbers are called

**components of the tensor**.

The abstract symbols of these components carry $k$ indexes where each of them runs from $n = 0$ to $n = 3$. For our purposes here only $k = 0, 1, 2$ are interesting.

(1) **Tensor of zeroth rank= four-scalar**
This tensor has $4^0 = 1$ component (*world-scalar*). It is a single quantity which remains invariant under a Lorentz transformation. An example is the length square $s^2$.

(2) **Tensor of first rank = four-vector**
This tensor possesses $4^1 = 4$ components. One distinguishes two types of vectors (*world-vectors*):

(2a) **Contravariant four-vector**
We mark this type of vector by superscripts:

$$a^{\mu} \equiv \left(a^0, a^1, a^2, a^3\right) \ . \tag{2.1}$$

The components transform with the change of the inertial system ($x^{\mu} \to x'^{\mu}$) as follows:

$$a'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} a^{\lambda} \ . \tag{2.2}$$

Since the change of coordinates shall be due to a Lorentz transformation it holds in particular:

$$a'^{\mu} = L_{\mu\lambda}\, a^{\lambda} \ . \tag{2.3}$$

Examples are

($\alpha$) the *position vector* $x^{\mu} \equiv (c\,t, x, y, z)$,
($\beta$) the differential $dx^{\mu}$, because for this the chain rule works:

$$dx'^{\mu} = \sum_{\lambda=0}^{3} \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\, dx^{\lambda} \ .$$

(2b) **Covariant four-vector**
This type of four-vector is marked by subscripts:

$$b_{\mu} = (b_0, b_1, b_2, b_3) \ . \tag{2.4}$$

The components transform as follows:

$$b'_{\mu} = \frac{\partial x^{\lambda}}{\partial x'^{\mu}}\, b_{\lambda} \ . \tag{2.5}$$

This means in the special case of the Lorentz transformation:

$$b'_\mu = \left(L^{-1}\right)_{\lambda\mu} \, b_\lambda \; . \tag{2.6}$$

The gradient of a scalar function $\varphi$ is an important example:

$$\begin{aligned} b_\mu &= \left(\frac{\partial\varphi}{\partial x^0}, \ldots, \frac{\partial\varphi}{x^3}\right) \; , \\ b'_\mu &= \left(\frac{\partial\varphi}{\partial x'^0}, \ldots, \frac{\partial\varphi}{\partial x'^3}\right) \; , \\ x^\nu &= x^\nu\left(x'^\mu\right) \; . \end{aligned} \tag{2.7}$$

It obviously holds:

$$b'_\mu = \frac{\partial\varphi}{\partial x'^\mu} = \frac{\partial\varphi}{\partial x^\nu}\,\frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\, b_\nu \; .$$

This corresponds to the definition equation (2.5).

(3) **Tensor of second rank**

This type of tensor possesses $4^2 = 16$ components. One distinguishes now three kinds of so-called *world-tensors*:

(3a) **contravariant tensor**

The components $F^{\alpha\beta}$ change under a Lorentz transformation as follows:

$$\left(F^{\mu\nu}\right)' = \frac{\partial x'^\mu}{\partial x^\alpha}\,\frac{\partial x'^\nu}{\partial x^\beta}\, F^{\alpha\beta} \; , \tag{2.8}$$

$$\left(F^{\mu\nu}\right)' = L_{\mu\alpha}\, L_{\nu\beta}\, F^{\alpha\beta} \; . \tag{2.9}$$

We see that ‘rows’ and ‘columns’ transform like contravariant vectors. As an example we cite the **tensor product** of two contravariant four-vectors $a^\mu$ and $b^\mu$, which consists of a total of 16 numbers (components):

$$F^{\mu\nu} = a^\mu\, b^\nu \; ; \quad \mu,\, \nu = 0, \ldots, 3 \; . \tag{2.10}$$

This tensor product transforms as:

$$\left(F^{\mu\nu}\right)' = a'^\mu\, b'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha}\,\frac{\partial x'^\nu}{\partial x^\beta} a^\alpha\, b^\beta = L_{\mu\alpha}\, L_{\nu\beta}\, F^{\alpha\beta} \; .$$

(3b) **covariant tensor**

That is now a system of 16 components $F_{\alpha\beta}$ which transform according to

$$F'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} F_{\alpha\beta} \ , \tag{2.11}$$

$$F'_{\mu\nu} = \left(L^{-1}\right)_{\alpha\mu} \left(L^{-1}\right)_{\beta\nu} F_{\alpha\beta} \tag{2.12}$$

In this case 'rows' and 'columns' transform like covariant vectors. The tensor product of two covariant four-vectors is an obvious example.

(3c) **mixed tensor**

The 16 components $F_\alpha^\beta$ transform in this case in the following manner

$$\left(F_\mu^\nu\right)' = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\beta} F_\alpha^\beta \ , \tag{2.13}$$

$$\left(F_\mu^\nu\right)' = \left(L^{-1}\right)_{\alpha\mu} L_{\nu\beta} F_\alpha^\beta \ . \tag{2.14}$$

We see that second-rank tensors can always be written as matrices. The elements of a normal matrix, though, do not necessarily transform like the components of a tensor. The formula (2.14), however, corresponds to the relation of the linear algebra,

$$F' = S^{-1} F S \ ,$$

which indicates how a matrix $F$ changes under a coordinate transformation into a matrix $F'$. The **mixed** tensor of second rank is therefore indeed a matrix in the strict sense, the covariant and contravariant tensors are not.

The tensor product of a covariant and a contravariant four-vector represents an example for a mixed tensor of second rank:

$$F_\mu^\nu = a^\nu b_\mu \ .$$

Completely analogously one defines tensors of still higher rank. For instance, in the example

$$\left(F^\mu_{\ \nu\rho\sigma}\right)' = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial x^\delta}{\partial x'^\sigma} F^\alpha_{\ \beta\gamma\delta}$$

the transformation behavior of a mixed tensor of fourth rank is defined. For our purposes here, however, only $k = 0, 1, 2$-tensors are relevant.

### 2.1.2 Calculation Rules

Which mathematical principles are to be followed by the just introduced tensors?

1. One multiplies a tensor with a number by multiplying each component with this number.
2. Two tensors are summed component by component!
3. By the

   **contraction of a tensor**

   one understands the equalizing of a superscript and a subscript, which automatically involves a summation. The rank of the tensor reduces thereby from $k$ to $k-2$.

   **Examples**

   (a) We put in the above mixed tensor of fourth rank $\nu = \mu$:

   $$\left(F^{\nu}_{\ \nu\rho\sigma}\right)' = \frac{\partial x'^{\nu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x^{\gamma}}{\partial x'^{\rho}} \frac{\partial x^{\delta}}{\partial x'^{\sigma}} F^{\alpha}_{\ \beta\gamma\delta} = \frac{\partial x^{\beta}}{\partial x^{\alpha}} \frac{\partial x^{\gamma}}{\partial x'^{\rho}} \frac{\partial x^{\delta}}{\partial x'^{\sigma}} F^{\alpha}_{\ \beta\gamma\delta}$$
   $$= \frac{\partial x^{\gamma}}{\partial x'^{\rho}} \frac{\partial x^{\delta}}{\partial x'^{\sigma}} F^{\alpha}_{\ \alpha\gamma\delta}.$$

   This expression transforms like a covariant tensor of second rank, as the comparison with (2.11) demonstrates.

   (b) The **trace** of a matrix $F^{\mu}_{\nu}$ is defined as the sum of its diagonal elements:

   $$F^{\mu}_{\nu} \longrightarrow F^{\nu}_{\nu} \ .$$

   The result is a tensor of zeroth rank, i.e. a scalar. The trace of a matrix is thus invariant under coordinate transformations.

   (c) The contraction of a **tensor product**

   $$a^{\mu} b_{\nu} \longrightarrow a^{\nu} b_{\nu}$$

   results of course in a scalar ($k = 2 \to k = 0$). It is equivalent to the scalar product in orthogonal coordinates. One therefore defines for four-vectors:

   **scalar product**

   $$(b, a) \equiv b_{\alpha}\, a^{\alpha} \ . \tag{2.15}$$

   This quantity is, as a scalar, Lorentz invariant. That can easily be checked:

   $$\left(b',\, a'\right) = b'_{\alpha}\, a'^{\alpha} = \frac{\partial x^{\lambda}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\rho}}\, b_{\lambda}\, a^{\rho} = \frac{\partial x^{\lambda}}{\partial x^{\rho}}\, b_{\lambda}\, a^{\rho} = b_{\lambda}\, a^{\lambda} = (b,\, a) \ .$$

(d) As example ($\beta$) after (2.3) we got to know the differential $dx^\mu$ of the position vector in the Minkowski space as a special contravariant four-vector:

$$dx^\mu \equiv \left(dx^0, dx^1, dx^2, dx^3\right) = (c\,dt, dx, dy, dz)\ . \tag{2.16}$$

Therewith we build the Lorentz invariant *length square*,

$$\begin{aligned}(ds)^2 &= \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 \\ &= (c\,dt)^2 - dx^2 - dy^2 - dz^2\ ,\end{aligned} \tag{2.17}$$

and write in short:

$$(ds)^2 = \mu_{\alpha\beta}\, dx^\alpha\, dx^\beta\ . \tag{2.18}$$

The coefficients $\mu_{\alpha\beta}$ are the components of the metric tensor (see (2.86), Vol. 2), which in the Special Theory of Relativity is symmetric ($\mu_{\alpha\beta} = \mu_{\beta\alpha}$) and diagonal:

**covariant metric tensor**

$$\mu_{\alpha\beta} \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\ . \tag{2.19}$$

If one writes the invariant length square $(ds)^2$ as scalar product of $dx$ with itself,

$$(ds)^2 = (dx,\, dx) = dx_\alpha\, dx^\alpha\ , \tag{2.20}$$

then one obviously gets by comparison with (2.18):

$$dx_\alpha = \mu_{\alpha\beta}\, dx^\beta\ . \tag{2.21}$$

This suggests the definition of the **contravariant metric tensor** by the following ansatz:

$$dx^\alpha = \mu^{\alpha\beta}\, dx_\beta\ . \tag{2.22}$$

That means:

$$dx^\gamma = \mu^{\gamma\alpha}\, dx_\alpha = \mu^{\gamma\alpha}\, \mu_{\alpha\beta}\, dx^\beta\ .$$

But this can be correct only if

$$\mu^{\gamma\alpha}\,\mu_{\alpha\beta} = \delta^{\gamma}_{\beta} = \begin{cases} 1, \text{ if } \gamma = \beta\,, \\ 0 \ \ \text{otherwise}\,. \end{cases} \tag{2.23}$$

We read off from (2.19) that for the Special Theory of Relativity covariant and contravariant metric tensor are obviously identical:

$$\mu^{\alpha\beta} = \mu_{\beta\alpha} = \mu_{\alpha\beta}\ . \tag{2.24}$$

4. Without explicit proof we generalize (2.21) and (2.22), respectively, to a **prescription** how to transform covariant into contravariant tensors and vice versa. Let us speak of

   **raising and lowering an index**

$$D^{\cdots\alpha\cdots}_{\cdots} = \mu^{\alpha\beta}\,D^{\cdots}_{\cdots\beta\cdots}\ , \tag{2.25}$$

$$D^{\cdots}_{\cdots\alpha\cdots} = \mu_{\alpha\beta}\,D^{\cdots\beta\cdots}_{\cdots}\ . \tag{2.26}$$

In this way one can almost arbitrarily *'play'* with the positions of the indexes. Let us put the rule to a test:

$$\begin{aligned} D^{\cdots}_{\cdots\alpha\cdots} &= \mu_{\alpha\beta}\,D^{\cdots\beta\cdots}_{\cdots} = \mu_{\alpha\beta}\,\mu^{\beta\gamma}\,D^{\cdots}_{\cdots\gamma\cdots} \\ &= \delta^{\gamma}_{\alpha}\,D^{\cdots}_{\cdots\gamma\cdots} = D^{\cdots}_{\cdots\alpha\cdots}\ . \end{aligned}$$

In particular, we can now transform with the above prescription each contravariant four-vector

$$a^{\mu} \equiv \left(a^0, a^1, a^2, a^3\right) = \left(a^0, \mathbf{a}\right) \tag{2.27}$$

into the corresponding covariant four-vector:

$$a_{\mu} \equiv (a_0, a_1, a_2, a_3) = \left(a^0, -\mathbf{a}\right)\ . \tag{2.28}$$

That means for the scalar product (2.15):

$$(b, a) = b_{\alpha}\,a^{\alpha} = \mu_{\alpha\beta}\,b^{\beta}\,a^{\alpha} = b^0\,a^0 - \mathbf{b}\cdot\mathbf{a}\,. \tag{2.29}$$

The last summand represents the normal three-dimensional scalar product between the space components of the two four-vectors. Note that the scalar product can be written as the sum of the products of the corresponding components only when one combines the covariant components of one of the four-vectors with the contravariant components of the other four-vector.

*Examples*

$$s^2 = (x, x) = c^2 t^2 - \mathbf{r}^2 \,,$$
$$(ds)^2 = (dx,\, dx) = (c\, dt)^2 - (d\mathbf{r})^2 \,.$$

### 2.1.3 Differential Operators

We obtain the transformation property of the differential operators, which are so important for Theoretical Physics, by a direct application of the chain rule:

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\alpha}} \,; \quad x^{\alpha} = x^{\alpha}\left(x'^{\mu}\right) \,.$$

The differentiation with respect to the component of a contravariant four-vector thus transforms as the component of a covariant four-vector. This applies directly to the nabla-operator:

**Gradient**

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \,. \tag{2.30}$$

$\nabla$ is the *normal* three-dimensional gradient (see (1.269), Vol. 1). Using the general prescription (2.26) we find for the derivative with respect to a covariant component:

$$\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \,. \tag{2.31}$$

The **divergence** (see (1.278), Vol. 1) is, as the scalar product of, respectively, a covariant gradient with a contravariant four-vector and a contravariant gradient with a covariant four-vector, which is of course Lorentz invariant:

**Divergence**

$$\partial_{\mu} a^{\mu} \equiv \partial^{\mu} a_{\mu} = \frac{1}{c}\frac{\partial}{\partial t} a^0 + \nabla \cdot \mathbf{a} \,. \tag{2.32}$$

Eventually, an important operator, in particular for electrodynamics, is the

**d'Alembert operator $\square$**

(see (4.30), Vol. 3):

$$-\square \equiv \partial_{\mu}\partial^{\mu} \equiv \partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \,. \tag{2.33}$$

$\Delta = \nabla^2$ is the Laplace operator (see (1.282), Vol. 1). As a scalar product the d'Alembert operator, too, is Lorentz invariant.

## 2.2 Covariant Formulation of Classical Mechanics

Let us now reformulate the basic laws of classical mechanics in such a way that they become form-invariant under Lorentz transformations. For this purpose we have to present them in covariant four-dimensional form, i.e., all terms in such an equation must be four-tensors of the same rank. Furthermore, in the limit $v \ll c$ the *'familiar'* relationships should be reproduced.

### *2.2.1 Proper Time, World-Velocity*

As **world-line**, we have denoted in Sect. 1.5, the trajectory of a physical particle. It is about the totality of all **events**

$$x^{\mu} = (c\,t, x, y, z) \ ,$$

passed by the object in this space in course of time. Then $dx^{\mu}$ is the differential change along the world-line. The square of the differential length

$$(ds)^2 = dx^{\mu}\, dx_{\mu} = c^2(dt)^2 - (d\mathbf{r})^2 \tag{2.34}$$

is, as already found out, as a scalar product a *world-scalar*, i.e., a Lorentz invariant. But the same must then be valid also for the *'time-quantity'*

$$(d\tau)^2 = \frac{1}{c^2}(ds)^2 = (dt)^2 - \frac{1}{c^2}\,(d\mathbf{r})^2 \ , \tag{2.35}$$

since the velocity of light $c$ has according to the fundamental Postulate 1.3.2 in Sect. 1.3 the same value in all inertial systems. We understand the physical meaning of $d\tau$ as follows. Since $(d\tau)^2$ is invariant, we can choose for an interpretation just a special, *most useful* reference system. A proper choice could be, for instance, the *co-moving* inertial system in which the particle is momentarily *at rest*:

$$dx^{\mu\prime} \equiv \left(c\,dt', 0, 0, 0\right) \ . \tag{2.36}$$

It follows then for $d\tau$:

$$(d\tau)^2 = \frac{1}{c^2}\,dx'^{\mu}\,dx'_{\mu} = \left(dt'\right)^2 \ . \tag{2.37}$$

$d\tau$ thus corresponds to a time interval on a clock which is carried along, i.e., to an interval of the **proper time** discussed in Sect. 1.4.3. Since $d\tau$ as world-scalar is Lorentz invariant, the proper time interval will of course not change when we transform it into a system $\Sigma$ which is moving compared to the particle:

$$c^2(d\tau)^2 = dx^\mu\, dx_\mu = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$
$$= c^2(dt)^2\left(1 - \frac{v^2}{c^2}\right) .$$

The result agrees with the statement in Sect. 1.4.3 that the *'proper time always goes slow'*:

$$dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \gamma\, d\tau > d\tau \; . \qquad (2.38)$$

$v$ is the relative velocity of the particle within the system $\Sigma$.

Let us now come back to the term of the

**world-velocity $u^\mu$,**

which is logically defined by the displacement $dx^\mu$ of the particle in the Minkowski space within the proper time $d\tau$:

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \; . \qquad (2.39)$$

It is about a contravariant four-vector for which we can also write:

$$u^\mu = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}}\frac{dx^\mu}{dt} = \gamma\,\frac{dx^\mu}{dt}$$
$$\Longrightarrow u^\mu = \frac{1}{\sqrt{1-\beta^2}}\left(c, v_x, v_y, v_z\right) = \gamma(v)\,(c, \mathbf{v}) \; . \qquad (2.40)$$

The *norm* of $u^\mu$ is, as scalar product, Lorentz invariant having therewith a rather simple physical meaning:

$$u^\mu\, u_\mu = \gamma^2\left(c^2 - v^2\right) = c^2 \; . \qquad (2.41)$$

### *2.2.2 Force, Momentum, Energy*

The Newtonian law of inertia (see (2.42), Vol. 1),

$$F_i = m\frac{d}{dt}v_i\, ; \quad i = x, y, z \; , \qquad (2.42)$$

retains, as we now know, its validity only when the relative velocity is sufficiently small ($v \ll c$). It is therewith form-invariant under a Galilean transformation. We now look for the relativistic generalization of this law in the four-dimensional Minkowski space. Thereby we have of course to again require, as a boundary condition, that for $v \ll c$ all the relations for the space components reduce to the form (2.42).

Unfortunately, the space components of the required four-force can not simply be identified with the $F_i$. These do not exhibit the correct transformation behavior. This is of course not surprising since we have seen that the space components of the four-velocity $u^\mu$ in (2.40), for instance, are not the $v_i$, but rather they have to be multiplied by the factor $\gamma(v)$. It is, however, necessary that the space components of each four-vector transform under the ordinary three-dimensional rotations as the familiar space vectors. We know, though, that the transformation behavior with respect to rotations of an ordinary three-dimensional space vector does not change when the vector is multiplied by a scalar. The space components of the four-force we are looking for should therefore be products of the $F_i$ with suitable scalar functions of $\beta = v/c$, which reduce to 1 for $v \ll c$.

In order to get the relativistic generalization of Newton's law (2.42) we at first replace the velocity $\mathbf{v}$ by the four-velocity $u^\mu$,

$$\mathbf{v} \longrightarrow u^\mu \; ,$$

since only the space components of $u^\mu$ reduce to $v_i$ $\beta \ll 1$. Furthermore, we will replace on the right-hand side of Eq. (2.42) the time $t$ by the proper time $\tau$,

$$t \longrightarrow \tau \; ,$$

because only the proper time is Lorentz invariant. The term

$$\frac{d}{d\tau} u^\mu$$

has therewith the dimension of an acceleration and is a contravariant four-vector which transforms as $x^\mu$ (see Exercise 2.5.3). Finally we still consider the inertial mass $m$ of the particle to be a Lorentz invariant since only *'space and time'* are in the Special Theory of Relativity subject to a critical revision; however, not the *'matter'*. That leads us to the following **ansatz** for the relativistic generalization of the force equation (2.42):

$$m \frac{d}{d\tau} u^\mu = K^\mu \; . \tag{2.43}$$

The contravariant four-vector $K^\mu$ is called

**Minkowski force**

Both sides of the force equation are world-tensors of first rank so that the covariance with respect to Lorentz transformations is assured. However, we still have to determine explicitly the components of the Minkowski force $K^{\mu}$.

To their determination we recall the other version of the non-relativistic law of inertia:

$$F_i = \frac{d}{dt} p_i \,; \quad i = x, y, z \,. \tag{2.44}$$

This law requires the conservation of momentum if no external forces act on the particle. But this *Newtonian form* of the momentum conservation is not yet Lorentz invariant. We can fix, however, the space-part of the relativistic momentum by presuming a Lorentz invariant conservation of momentum for a force-free particle. For this purpose we bring the force equation for the space components into a form which is structurally rather similar to the law of inertia (2.44):

$$K_i = m \frac{d}{d\tau} u_i = m \, \gamma \frac{d}{dt} \gamma \, v_i \,; \quad i = x, y, z \,. \tag{2.45}$$

The conservation law of momentum is certainly Lorentz invariant when we fix by comparison the momenta and forces as follows:

$$p_{\mathrm{r}i} = \frac{m \, v_i}{\sqrt{1-\beta^2}} = \gamma \, m v_i \,, \tag{2.46}$$

$$K_i = \frac{F_i}{\sqrt{1-\beta^2}} = \gamma F_i \,, \tag{2.47}$$

$$i = x, y, z \,.$$

$F_i$ are now, differently to (2.44), the relativistic force components $F_i = \frac{d}{dt} p_{\mathrm{r}i}$. As required these expressions reduce for $v \ll c$ to the known, non-relativistic terms. By discussing particle collisions in Sect. 2.2.3 we will realize that (2.46) is probably the only conclusive relativistic generalization of the mechanical momentum.

The time component of the Minkowski force is still to be considered. To get it we calculate:

$$\begin{aligned} K^{\mu} \, u_{\mu} &= K^0 \, u^0 - \mathbf{K} \cdot \mathbf{u} = \left( m \frac{d}{d\tau} u^0 \right) u^0 - \left( m \frac{d}{d\tau} \mathbf{u} \right) \cdot \mathbf{u} \\ &= \frac{1}{2} m \, \frac{d}{d\tau} \left( u^0 \, u^0 - \mathbf{u} \cdot \mathbf{u} \right) = \frac{1}{2} m \, \frac{d}{d\tau} (u^{\mu} \, u_{\mu}) = \frac{1}{2} m \, \frac{d}{d\tau} c^2 \,. \end{aligned}$$

Here we have exploited (2.29) and (2.41). Hence we have

$$K^{\mu} \, u_{\mu} = 0 \,. \tag{2.48}$$

But on the other-hand it also holds with (2.40) and (2.47):

$$K^{\mu}\, u_{\mu} = \gamma K^0 c - \gamma^2\, \mathbf{F}\cdot\mathbf{v}\,. \tag{2.49}$$

The comparison with (2.48) gives us the zeroth force component:

$$K^0 = \gamma \frac{\mathbf{F}\cdot\mathbf{v}}{c}\,. \tag{2.50}$$

Equations (2.47) and (2.50) lead to the complete version of the

**Minkowski force**

$$K^{\mu} = \gamma \left( \frac{\mathbf{F}\cdot\mathbf{v}}{c}, F_x, F_y, F_z \right)\,. \tag{2.51}$$

The Newtonian law of inertia in the form of (2.43) is therewith relativistically generalized.

Next we investigate the physical meaning of the time component of the Minkowski force:

$$\gamma \frac{\mathbf{F}\cdot\mathbf{v}}{c} = m\,\frac{d}{d\tau}\,u^0 = m\,\gamma \frac{d}{dt}(\gamma\, c)$$
$$\Longrightarrow \mathbf{F}\cdot\mathbf{v} = \frac{d}{dt}\,\frac{m\,c^2}{\sqrt{1-\beta^2}}\,. \tag{2.52}$$

The scalar product $\mathbf{F}\cdot\mathbf{v}$ corresponds to the work which the force $\mathbf{F}$ exerts per time-unit on the particle with the mass $m$. In the non-relativistic mechanics this work is identical to the time change of the kinetic energy $T$ (see (2.226), Vol. 1). We therefore choose the ansatz

$$\mathbf{F}\cdot\mathbf{v} = \frac{d}{dt} T_{\mathrm{r}}\,, \tag{2.53}$$

where the index 'r' stands for *relativistic*. One obtains then by comparison with (2.52) the

**relativistic kinetic energy**

$$T_{\mathrm{r}} = \frac{m\,c^2}{\sqrt{1-v^2/c^2}} = m\gamma c^2\,. \tag{2.54}$$

In the limit of small velocities ($v \ll c$), we expect this quantity that also to reduce to the familiar non-relativistic expression $T = \frac{m}{2}v^2$. But the expansion,

$$T_{\mathrm{r}} = m\,c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = m\,c^2 + \frac{1}{2} m\,v^2 + \frac{3}{8}\,m\,\frac{v^4}{c^2} + \ldots\,,$$

shows that $T_\mathrm{r}$ does not actually reduce for small $v/c$ to the non-relativistic kinetic energy:

$$T_\mathrm{r} \underset{v/c \ll 1}{\longrightarrow} m\,c^2 + \frac{1}{2} m\,v^2 \,. \tag{2.55}$$

The *'disturbing'* additional term $m\,c^2$ is a constant which actually does not affect the kinematics of the mass point. One could, for instance, subtract it on the right-hand side of the definition equation (2.54) of $T_\mathrm{r}$, since the relativistic kinetic energy can be fixed by analogy from (2.53) to (2.54) anyway only up to an additive constant. Later we will see, however, that a more profound physical meaning has to be ascribed to this additive constant:

$$m\,c^2 \iff \textbf{rest energy of the mass point}\,.$$

We therefore retain it here.

By multiplying the four-velocity $u^\mu$ (2.40) with the mass $m$ of the particle, we can define a new contravariant four-vector which we interpret as

**Four-momentum (world-momentum)**

$$p^\mu = m\,u^\mu = m\gamma\,(c, \mathbf{v})\,. \tag{2.56}$$

Inserting (2.40) we get:

$$p^\mu = \left(\frac{T_\mathrm{r}}{c}, \gamma\,m\,v_x,\ \gamma\,m\,v_y,\ \gamma\,m\,v_z\right) \equiv \left(\frac{T_\mathrm{r}}{c}, \mathbf{p}_\mathrm{r}\right)\,. \tag{2.57}$$

The space components thus correspond to the relativistic generalization (2.46) of the mechanical momentum-vector $\mathbf{p} = m\,\mathbf{v}$,

$$\mathbf{p}_\mathrm{r} = \gamma\,\mathbf{p} = \frac{m}{\sqrt{1 - v^2/c^2}}\,\mathbf{v}\,, \tag{2.58}$$

while the time component is essentially identical to the kinetic energy.

In the so far pursued line of argument the mass $m$ is a scalar invariant property of the particle. However, in all the important formulae $m$ always appears in the combination

$$m(v) = \gamma\,(v)\,m = \frac{m}{\sqrt{1 - v^2/c^2}}, \tag{2.59}$$

which is therefore sometimes interpreted as *'velocity dependent relativistic mass'*. The space components of the world-momentum,

$$\mathbf{p}_\mathrm{r} = m(v)\,\mathbf{v}\,, \tag{2.60}$$

then have formally the same form as in the non-relativistic mechanics. Strictly speaking, that is the only reason for the introduction of $m(v)$. The symbol $m$ without argument then stands for $m(0)$ and means the **'rest mass'** of the particle. The relativistic kinetic energy $T_\mathrm{r}$ (2.54), too, can be written with $m(v)$ in a simpler way:

$$T_\mathrm{r} = m(v)\, c^2 \; . \tag{2.61}$$

Since $m(v)$ merely represents an abbreviated notation we will **not** make use of the definition (2.59). It must be considered rather as somewhat inconvenient as it clouds the fact that 'mass' as the direct measure of the 'amount of matter' should be independent of the system of coordinates.

The norm of the four-momentum,

$$p^\mu p_\mu = \frac{T_\mathrm{r}^2}{c^2} - \mathbf{p}_\mathrm{r}^2 = m^2 u^\mu u_\mu = m^2 c^2 \; , \tag{2.62}$$

as a scalar product is of course Lorentz invariant. We have therewith found for the

**relativistic energy of a free particle**

an alternative representation:

$$T_\mathrm{r} = E = \sqrt{c^2 \mathbf{p}_\mathrm{r}^2 + m^2 c^4} \; . \tag{2.63}$$

Because of the equivalence principle (Sect. 1.3) we have to conclude that

$$\textbf{conservation of momentum} : \quad \mathbf{p}_\mathrm{r} = \gamma\, m\, \mathbf{v} = \textbf{const} \tag{2.64}$$

must be valid for a force-free motion in all inertial systems, i.e., it does not depend on the choice of the reference system. But $\mathbf{p}_\mathrm{r}$ consists of the three space components of the four-vector $p^\mu$. Hence we have to conclude:

$$\mathbf{p}_\mathrm{r} = \textbf{const} \implies T_\mathrm{r} = \mathrm{const} \; . \tag{2.65}$$

For, when one transforms the contravariant four-vector $p^\mu$ according to (2.2) to another inertial system $\Sigma'$, *new* space components $\mathbf{p}_\mathrm{r}$ will come out, which also depend on $T_\mathrm{r}$. Were $T_\mathrm{r} \neq$ const, then, consequently, the conservation of momentum in $\Sigma'$ would **no longer** be valid:

**momentum conservation $\Longleftrightarrow$ energy conservation.**

In the Special Theory of Relativity via the world-momentum $p^\mu$, there does exist a close connection between these two laws of conservation.

From this result we draw a very important conclusion. As is well-known, $\mathbf{p} =$ **const** is non-relativistically possible even if $T \neq$ const. Think of an exploding grenade. The momenta of the fragments add together vectorially to the constant which describes the momentum **before** the explosion, while the kinetic energy, as one knows, changes dramatically. The **relativistic** kinetic energy, in contrast, **cannot** have changed! But because of $T_r \approx m c^2 + T$ (see (2.55)), that is only possible if a change of the rest energy compensates the change of $T$ during the process. Since the velocity of light $c$ is a universal constant this leads to

**Einstein's equivalence of mass and energy**

$$\Delta E = \Delta m c^2 . \tag{2.66}$$

Let us illustrate the important meaning of this relation by some examples:

1. *mass increase.*
   if one lifts 100 kg by 1 km upwards:

   $$\Delta m = 10^{-10}\,\text{kg} .$$

2. *pair production*:
   The decay of a mass-less photon $\nu$ into an electron ($e^-$) and a positron ($e^+$) is possible provided that

   $$E_\nu \geq 2\, m_e c^2 = 1.022\,\text{MeV} .$$

   The energy difference,

   $$E_\nu - 2\, m_e c^2 = T\,(e^-) + T\left(e^+\right) ,$$

   appears as the sum of the kinetic energies of the electron and the positron. The reversal (*pair annihilation*),

   $$e^+ + e^- \to \nu ,$$

   is of course also possible.
3. *Mass-loss of the sun due to energy radiation*:

   $$\frac{\Delta m}{\Delta t} \approx 4 \cdot 10^{12}\,\frac{\text{kg}}{\text{s}} .$$

4. *atomic bomb*
   The total momentum remains unchanged after the explosion. It results, however, in the release of a *terribly high* kinetic energy of the fragments due to a mass-loss of about 0.1 %.
5. *nuclear fission, nuclear fusion*.

### 2.2.3 *Elastic Collision*

In the last section the relativistic form $\mathbf{p}_\mathrm{r}$ of the mechanical momentum has been introduced more or less by analogy considerations. That was also valid for the relativistic kinetic energy $T_\mathrm{r}$. We now try to perform a more direct derivation of these quantities with the **assumption** of

**momentum and energy conservation in closed inertial systems**

The focus is of course on the **relativistic** energies and momenta which are still to be derived. The familiar non-relativistic momentum conservation law, for instance, is, as we know, **not** Lorentz invariant. We start with the following ansatz,

$$\mathbf{p}_\mathrm{r} = m(v)\mathbf{v} \; ; \quad T_\mathrm{r} = \varepsilon(v) \; , \tag{2.67}$$

where $m(v)$, as explained in connection with (2.61), is to be understood as an abbreviation. The same also holds for $\varepsilon(v)$. $\varepsilon(v)$ as well as $m(v)$ are to be considered at first as unknowns which must fulfill the boundary conditions

$$m(0) = m \; ; \quad \frac{d\varepsilon}{dv^2}(0) = \frac{m}{2} \; . \tag{2.68}$$

They will be derived by inspecting the

**elastic collision of two identical particles.**

We can certainly assume that $\varepsilon$ is a monotonic function of $v$. Let us first discuss the collision in the

**center-of-mass system** $\Sigma'$

of the two particles. We denote:

$$\begin{aligned} &\mathbf{v}'_a, \mathbf{v}'_b : \quad \text{velocities } \textit{before} \text{ the collision} \; , \\ &\mathbf{v}'_c, \mathbf{v}'_d : \quad \text{velocities } \textit{after} \text{ the collision} \; . \end{aligned}$$

In the center-of-mass system it must of course be

$$\mathbf{v}'_a = \mathbf{u} \; ; \quad \mathbf{v}'_b = -\mathbf{u} \; ,$$

where we can assume, without loss of generality, that $\mathbf{u}$ defines the direction of the $z$-axis in $\Sigma'$. From the **energy law** in $\Sigma'$,

$$2\,\varepsilon(u) = \varepsilon\left(v'_c\right) + \varepsilon\left(v'_d\right) \; ,$$

it must follow $\varepsilon(v'_c) = \varepsilon(v'_d)$, since the particles are assumed to be identical. But because of the monotony of $\varepsilon(v)$ this means also $v'_c = v'_d = u$. Therefore, all

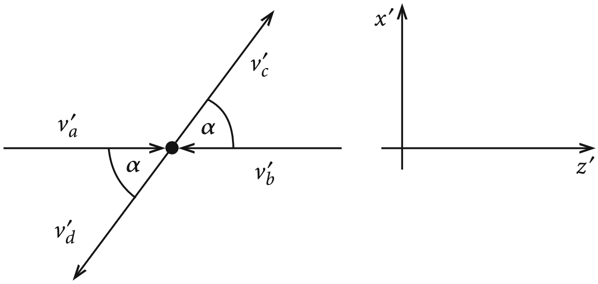



**Fig. 2.1** Velocities for the elastic collision

the four velocities have the same magnitudes. From the **law of conservation of momentum** in $\Sigma'$,

$$m(u)\mathbf{u} - m(u)\mathbf{u} = m(u)\left(\mathbf{v}'_c + \mathbf{v}'_d\right) \ ,$$

we can read off:

$$\mathbf{v}'_c = -\mathbf{v}'_d$$

With the notations used in Fig. 2.1, that yields:

$$\mathbf{v}'_a = (0, 0, u) \ ; \quad \mathbf{v}'_b = (0, 0, -u) \ ,$$
$$\mathbf{v}'_c = (u \sin\alpha, 0, u\cos\alpha) \ ; \quad \mathbf{v}'_d = (-u\sin\alpha, 0, -u\cos\alpha) \ .$$

The angle $\alpha$ remains undetermined.

We now perform the analogous considerations for the inertial system $\Sigma$, which moves relative to $\Sigma'$ with the velocity $(-\mathbf{u})$. For the particle velocities in $\Sigma$ we apply the transformation formulae (1.39) to (1.41):

$$\mathbf{v}_a = \left(0, 0, \frac{2u}{1+\frac{u^2}{c^2}}\right) \ ,$$
$$\mathbf{v}_b = (0, 0, 0) \ ,$$
$$\mathbf{v}_c = \left(\frac{1}{\gamma}\,\frac{u\sin\alpha}{1+\frac{u^2}{c^2}\cos\alpha},\ 0,\ \frac{u(1+\cos\alpha)}{1+\frac{u^2}{c^2}\cos\alpha}\right) \ ,$$
$$\mathbf{v}_d = \left(\frac{1}{\gamma}\,\frac{-u\sin\alpha}{1-\frac{u^2}{c^2}\cos\alpha},\ 0,\ \frac{u(1-\cos\alpha)}{1-\frac{u^2}{c^2}\cos\alpha}\right) \ ,$$
$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-(1/2)} \ .$$

The law of conservation of momentum, which we have already used for $\Sigma'$ and which is valid, as per presumption, in all inertial systems,

$$m\,(v_a)\,\mathbf{v}_a + m\,(v_b)\,\mathbf{v}_b = m\,(v_c)\,\mathbf{v}_c + m\,(v_d)\,\mathbf{v}_d\ ,$$

must be fulfilled for each component, thus in particular for the $x$-component:

$$0 = \frac{u}{\gamma}\left(m\,(v_c)\,\frac{\sin\alpha}{1+\frac{u^2}{c^2}\cos\alpha} - m\,(v_d)\,\frac{\sin\alpha}{1-\frac{u^2}{c^2}\cos\alpha}\right)\ .$$

It follows:

$$m\,(v_c) = \frac{1+\frac{u^2}{c^2}\cos\alpha}{1-\frac{u^2}{c^2}\cos\alpha}\,m\,(v_d)\ . \tag{2.69}$$

This formula must be valid for **all** scattering angles $\alpha$, therefore also for $\alpha \to 0$. In this special case, however,

$$\mathbf{v}_c \approx \mathbf{v}_a \quad \text{and} \quad \mathbf{v}_d \approx 0\ ,$$

so that (2.69) reads

$$m\,(v_a) = \frac{1+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}m(0)\ . \tag{2.70}$$

We reformulate the prefactor:

$$\left(\frac{1-\frac{u^2}{c^2}}{1+\frac{u^2}{c^2}}\right)^2 = \frac{\left(1+\frac{u^2}{c^2}\right)^2 - 4\frac{u^2}{c^2}}{\left(1+\frac{u^2}{c^2}\right)^2} = 1 - \frac{1}{c^2}\,\frac{4\,u^2}{\left(1+\frac{u^2}{c^2}\right)^2} = 1 - \frac{v_a^2}{c^2} = \frac{1}{\gamma_a^2}\ .$$

This yields in (2.70) $m(v_a) = \gamma_a\,m(0)$. We can now drop the index $a$ and take $m(0) = m$ corresponding to the boundary condition (2.68):

$$m(v) = \frac{m}{\sqrt{1-v^2/c^2}}\ . \tag{2.71}$$

This is exactly the previous result (2.59). By insertion of this relation into the ansatz (2.67) we obtain the required relativistic momentum of a particle of the mass $m$ and the velocity $\mathbf{v}$:

$$\mathbf{p}_\mathrm{r} = \frac{m}{\sqrt{1-v^2/c^2}}\mathbf{v}\ . \tag{2.72}$$

This is indeed the expression (2.58) which we found previously by analogy-arguments.

Now we further want to determine the kinetic energy $T_\mathrm{r} = \varepsilon(v)$ using the theory of the elastic collision of the two identical particles. It holds in the inertial system $\Sigma$ since the particle $b$ is *'at rest'* before the collision:

$$\varepsilon\left(v_a\right) + \varepsilon(0) = \varepsilon\left(v_c\right) + \varepsilon\left(v_d\right) \ .$$

We investigate again the case $\alpha \to 0$, which affects only the right-hand side of the energy equation. There appear only the magnitudes of the velocities $\mathbf{v_c}$ and $\mathbf{v}_d$, for which one finds:

$$v_c = \frac{1}{1 + \frac{u^2}{c^2}\cos\alpha}\sqrt{\frac{u^2}{\gamma^2}\sin^2\alpha + u^2(1+\cos\alpha)^2}$$

$$v_d = \frac{1}{1 - \frac{u^2}{c^2}\cos\alpha}\sqrt{\frac{u^2}{\gamma^2}\sin^2\alpha + u^2(1-\cos\alpha)^2}$$

One recognizes that $v_c$ and $v_d$ are even functions of $\alpha$. Series expansions with respect to the powers of $\alpha$, which surely converge for $\alpha \to 0$, contain exclusively even powers of $\alpha$. This transfers to the energies $\varepsilon(v_c)$ and $\varepsilon(v_d)$, which we can expand therefore in powers of $\alpha^2$:

$$\begin{aligned}\varepsilon\left(v_a\right) + \varepsilon(0) = \ & \left.\varepsilon\left(v_c\right)\right|_{\alpha=0} + \alpha^2\left.\left(\frac{d\varepsilon\left(v_c\right)}{dv_c^2}\,\frac{dv_c^2}{d\alpha^2}\right)\right|_{\alpha=0} \\ & + \left.\varepsilon\left(v_d\right)\right|_{\alpha=0} + \alpha^2\left.\left(\frac{d\varepsilon\left(v_d\right)}{dv_d^2}\,\frac{dv_d^2}{d\alpha^2}\right)\right|_{\alpha=0} + \mathcal{O}\left(\alpha^4\right) \ .\end{aligned}$$

Because of

$$\mathbf{v}_c^2(\alpha = 0) = \mathbf{v}_a^2\ ; \quad \mathbf{v}_d^2(\alpha = 0) = \mathbf{v}_b^2 = 0$$

it is left to be analyzed:

$$0 \overset{!}{=} \left.\left(\frac{d\varepsilon\left(v_c\right)}{dv_c^2}\,\frac{dv_c^2}{d\alpha^2}\right)\right|_{\alpha=0} + \left.\left(\frac{d\varepsilon\left(v_d\right)}{dv_d^2}\,\frac{dv_d^2}{d\alpha^2}\right)\right|_{\alpha=0} \ .$$

Finally, we can still exploit the boundary condition (2.68):

$$0 = \frac{d\varepsilon\left(v_a\right)}{dv_a^2}\left(\frac{dv_c^2}{d\alpha^2}\right)_{\alpha=0} + \frac{m}{2}\left(\frac{dv_d^2}{d\alpha^2}\right)_{\alpha=0} \ . \tag{2.73}$$

To proceed we have to now expand the velocity squares $v_c^2$ and $v_d^2$ in $\alpha^2$:

$$
\begin{aligned}
v_d^2 &= \left(1 - \frac{u^2}{c^2}\cos\alpha\right)^{-2} \left[\frac{u^2}{\gamma^2}\sin^2\alpha + u^2(1-\cos\alpha)^2\right] \\
&= \left(1 - \frac{u^2}{c^2} + \frac{1}{2}\frac{u^2}{c^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right)\right)^{-2} \left(\frac{u^2}{\gamma^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right)\right) \\
&= \gamma^4 \left(1 + \frac{1}{2}\frac{u^2}{c^2}\alpha^2\gamma^2 + \mathcal{O}\left(\alpha^4\right)\right)^{-2} \left(\frac{u^2}{\gamma^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right)\right) \\
&= \gamma^4 \left(1 - \frac{u^2}{c^2}\alpha^2\gamma^2 + \mathcal{O}\left(\alpha^4\right)\right) \left(\frac{u^2}{\gamma^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right)\right) \\
&= \gamma^2 u^2 \alpha^2 + \mathcal{O}\left(\alpha^4\right) \ .
\end{aligned}
$$

From this it follows:

$$
\left(\frac{dv_d^2}{d\alpha^2}\right)_{\alpha=0} = \gamma^2\, u^2 = \frac{u^2}{1 - \frac{u^2}{c^2}} \ . \tag{2.74}
$$

We used for the above expansion

$$
(1+x)^{n/m} = 1 + \frac{n}{m}x + \mathcal{O}\left(x^2\right) \ . \tag{2.75}
$$

This formula helps us also when expanding $v_c^2$:

$$
\begin{aligned}
v_c^2 &= \left(1 + \frac{u^2}{c^2}\cos\alpha\right)^{-2} \left[\frac{u^2}{\gamma^2}\sin^2\alpha + u^2(1+\cos\alpha)^2\right] \\
&= \left(1 + \frac{u^2}{c^2} - \frac{1}{2}\,\frac{u^2}{c^2}\alpha^2 + \mathcal{O}\left(\alpha^4\right)\right)^{-2} \left[\frac{u^2}{\gamma^2}\alpha^2 + u^2\left(2 - \frac{1}{2}\alpha^2\right)^2 + \mathcal{O}\left(\alpha^4\right)\right] \\
&= \left(1 + \frac{u^2}{c^2}\right)^{-2} \left(1 - \frac{1}{2}\,\frac{u^2}{c^2}\,\frac{\alpha^2}{1 + \frac{u^2}{c^2}} + \mathcal{O}\left(\alpha^4\right)\right)^{-2} \\
&\qquad \cdot \left(\frac{u^2}{\gamma^2}\alpha^2 + 4u^2 - 2\alpha^2 u^2 + \mathcal{O}\left(\alpha^4\right)\right) \\
&= \frac{v_a^2}{4u^2}\left(1 + \frac{u^2}{c^2}\,\frac{\alpha^2 v_a}{2u} + \mathcal{O}\left(\alpha^4\right)\right)\left[4u^2 + u^2\alpha^2\left(\frac{1}{\gamma^2} - 2\right) + \mathcal{O}\left(\alpha^4\right)\right] \\
&= \frac{v_a^2}{4u^2}\left\{4u^2 + \alpha^2\left[\frac{2u^3}{c^2}v_a + u^2\left(\frac{1}{\gamma^2} - 2\right)\right] + \mathcal{O}\left(\alpha^4\right)\right\} \ .
\end{aligned}
$$

This leads to:

$$\left(\frac{dv_c^2}{d\alpha^2}\right)_{\alpha=0} = \frac{1}{4}v_a^2\left(\frac{2u}{c^2}v_a + \frac{1}{\gamma^2} - 2\right) = \frac{1}{4}v_a^2\left(\frac{4\frac{u^2}{c^2}}{1+\frac{u^2}{c^2}} - \frac{u^2}{c^2} - 1\right)$$

$$= -\frac{1}{4}v_a^2\frac{\left(1-\frac{u^2}{c^2}\right)^2}{1+\frac{u^2}{c^2}} = -\frac{1}{4}v_a^2\left(1+\frac{u^2}{c^2}\right)\left(\frac{1-\frac{u^2}{c^2}}{1+\frac{u^2}{c^2}}\right)^2 .$$

The last factor we have already evaluated in connection with (2.70):

$$\left(\frac{dv_c^2}{d\alpha^2}\right)_{\alpha=0} = -\frac{1}{4}\frac{v_a^2}{\gamma_a^2}\left(1+\frac{u^2}{c^2}\right) . \tag{2.76}$$

We insert (2.76) and (2.74) into (2.73):

$$\frac{d\varepsilon\,(v_a)}{dv_a^2} = \frac{m}{2}\,\frac{u^2}{1-\frac{u^2}{c^2}}\,\frac{4\gamma_a^2}{v_a^2\left(1+\frac{u^2}{c^2}\right)} = \frac{m}{2}\,\gamma_a^2\,\frac{1+\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}}$$

$$= \frac{m}{2}\,\gamma_a^3 = m\,c^2\,\frac{d}{dv_a^2}\left(1-\frac{v_a^2}{c^2}\right)^{-1/2} .$$

When we integrate this expression and drop from now on the index $a$ we are finally left with:

$$T_{\mathrm{r}} = \varepsilon(v) = \frac{m\,c^2}{\sqrt{1-v^2/c^2}} + d\,. \tag{2.77}$$

Except for the constant $d$,

$$d = \varepsilon(0) - m\,c^2\,, \tag{2.78}$$

we could reproduce by an analysis of the elastic collision the relativistic energy of the free particle (2.54) which we found in the last section more or less by analogy-arguments. At the end of this section we will explicitly prove that $d = 0$ and therewith necessarily

$$\varepsilon(0) = m\,c^2 \quad (\textit{rest energy}) \tag{2.79}$$

In order to avoid unnecessary paperwork let us put right now $d = 0$ for the following considerations.

According to the line of argument, followed in this section, we do not yet know at this point anything about a four-vector $p^{\mu}$. It is therefore an interesting question to ask how energy $T_\mathrm{r}$ and momentum $\mathbf{p}_\mathrm{r}$ behave under a Lorentz transformation:

$$\Sigma \xrightarrow{v} \Sigma' \; ; \qquad \gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \; .$$

In $\Sigma$ the particle may have the velocity

$$\mathbf{u} = \left(u_x, u_y, u_z\right) \; ,$$

the relativistic momentum

$$\mathbf{p}_\mathrm{r} = \left(p_\mathrm{rx},\, p_\mathrm{ry},\, p_\mathrm{rz}\right) = m\,\gamma_u\left(u_x, u_y, u_z\right) \; , \qquad \gamma_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

and the relativistic energy:

$$T_\mathrm{r} = m\,c^2\gamma_u \; .$$

The corresponding *'primed'* quantities $\mathbf{u}', \mathbf{p}'_\mathrm{r},\, T'_\mathrm{r}$ point out the properties of the particle in $\Sigma'$. For the transition $\mathbf{u} \rightarrow \mathbf{u}'$, $\mathbf{p}_\mathrm{r} \rightarrow \mathbf{p}'_\mathrm{r}$ and $T_\mathrm{r} \rightarrow T'_\mathrm{r}$ we use again the transformation formulae (1.39) to (1.41):

$$u'_{x,y} = \frac{1}{\gamma_v}\,\frac{u_{x,y}}{1 - \dfrac{v\,u_z}{c^2}} \; ; \quad u'_z = \frac{u_z - v}{1 - \dfrac{v\,u_z}{c^2}} \; .$$

Therewith we calculate at first $\gamma'_u$:

$$\begin{aligned}
u'^2 &= \left(1 - \frac{v\,u_z}{c^2}\right)^{-2}\left[\frac{1}{\gamma_v^2}\left(u_x^2 + u_y^2\right) + (u_z - v)^2\right] \\
&= \left(1 - \frac{v\,u_z}{c^2}\right)^{-2}\left[\left(1 - \frac{v^2}{c^2}\right)\left(u^2 - u_z^2\right) + u_z^2 + v^2 - 2v\,u_z\right] \\
&= \frac{c^2}{\left(c - \dfrac{v\,u_z}{c}\right)^2}\left(u^2 - \frac{v^2u^2}{c^2} + \frac{v^2u_z^2}{c^2} + v^2 - 2v\,u_z\right) \\
&= \frac{c^2}{\left(c - \dfrac{v\,u_z}{c}\right)^2}\left[\frac{u^2}{\gamma_v^2} + \left(c - \frac{v\,u_z}{c}\right)^2 - \frac{c^2}{\gamma_v^2}\right] \\
&= c^2 + \frac{1}{\left(1 - \dfrac{v\,u_z}{c^2}\right)^2}\,\frac{1}{\gamma_v^2}(u^2 - c^2) \; .
\end{aligned}$$

From that it follows with

$$1 - \frac{u'^2}{c^2} = \frac{1}{\left(1 - \frac{v\,u_z}{c^2}\right)^2} \frac{1}{\gamma_v^2} \frac{1}{\gamma_u^2}$$

the required expression for $\gamma'_u$:

$$\gamma'_u = \gamma_u \gamma_v \left(1 - \frac{v\,u_z}{c^2}\right) \ . \tag{2.80}$$

The transformed momenta are now easily determinable:

$$\begin{aligned} p'_{\mathrm{r}x,y} &= m\,\gamma'_u u'_{x,y} = m\,\gamma_u \gamma_v \left(1 - \frac{v\,u_z}{c^2}\right) \frac{1}{\gamma_v} \frac{u_{x,y}}{1 - \frac{v\,u_z}{c^2}} \\ &= m\,\gamma_u u_{x,y} = p_{\mathrm{r}x,y}, \\ p'_{\mathrm{r}z} &= m\,\gamma'_u u'_z = m\,\gamma_u \gamma_v \left(1 - \frac{v\,u_z}{c^2}\right) \frac{u_z - v}{1 - \frac{v\,u_z}{c^2}} \\ &= \gamma_v \left(m\,\gamma_u u_z - m\,\gamma_u v\right) = \gamma_v \left(p_{\mathrm{r}z} - v \frac{T_\mathrm{r}}{c^2}\right) \ . \end{aligned}$$

Just as easily we find with (2.80) the transformed energy:

$$\begin{aligned} T'_\mathrm{r} &= m\,c^2 \gamma'_u = m\,c^2 \gamma_u \gamma_v \left(1 - \frac{v\,u_z}{c^2}\right) \\ &= \gamma_v \left(m\,c^2 \gamma_u - m\,\gamma_u u_z v\right) = \gamma_v \left(T_\mathrm{r} - v\,p_{\mathrm{r}z}\right) \ . \end{aligned}$$

Let us compile the results:

$$\begin{aligned} \frac{T'_\mathrm{r}}{c} &= \gamma_v \left(\frac{T_\mathrm{r}}{c} - \beta\,p_{\mathrm{r}z}\right) \\ p'_{\mathrm{r}x} &= p_{\mathrm{r}x} \\ p'_{\mathrm{r}y} &= p_{\mathrm{r}y} \\ p'_{\mathrm{r}z} &= \gamma_v \left(p_{\mathrm{r}z} - \beta \frac{T_\mathrm{r}}{c}\right) \ . \end{aligned} \tag{2.81}$$

When we perceive these four quantities as components of the vector $p^\mu$ then we recognize that they transform like the components of a contravariant four-vector:

$$\begin{aligned} p^\mu &= \left(p^0, p^1, p^2, p^3\right) = \left(\frac{T_\mathrm{r}}{c}, p_{\mathrm{r}x}, p_{\mathrm{r}y}, p_{\mathrm{r}z}\right) \\ &= \gamma_u m \left(c, u_x, u_y, u_z\right) = m\,u^\mu \ . \end{aligned} \tag{2.82}$$

For the components of the transformed four-vector $p^{\mu\prime}$ we have with (2.16) and (2.81) :

$$p^{\mu\prime} = L_{\mu\,\lambda}\, p^{\lambda}\ . \tag{2.83}$$

This, however, is the definition equation (2.3) of a contravariant four-vector. Therewith now, we have explicitly derived the four-vector *world-momentum*, which we had introduced already in (2.57) by analogy-arguments.

The next item concerns the **transformation of forces**. For the space components we need the time derivatives of relativistic momenta:

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}_{\mathrm{r}}\ ; \quad \mathbf{F}' = \frac{d}{dt'}\mathbf{p}'_{\mathrm{r}}\ .$$

Let $\mathbf{u}$ and $\mathbf{u}'$ further on be the particle velocities in $\Sigma$ and $\Sigma'$, respectively, where $\Sigma'$ moves relative to $\Sigma$ with the velocity $v$ parallel to the $z$-axis. According to (1.21) the time transforms as:

$$t' = \gamma_v \left(t - \frac{v}{c^2}z\right)\ .$$

If we exploit (2.80), this means for the time-differential:

$$dt' = \gamma_v\, dt \left(1 - \frac{v\, u_z}{c^2}\right) = \frac{\gamma'_u}{\gamma_u}dt\ .$$

From this we read off,

$$\frac{d}{dt'} \equiv \frac{\gamma_u}{\gamma'_u}\frac{d}{dt}\ , \tag{2.84}$$

which immediately leads to the transformed forces:

$$F'_x = \frac{d}{dt'}p'_{\mathrm{r}x} = \frac{\gamma_u}{\gamma'_u}\frac{d}{dt}p_{\mathrm{r}x} = \frac{\gamma_u}{\gamma'_u}F_x\ .$$

Here we have taken from (2.81) $p'_{\mathrm{r}x} = p_{\mathrm{r}x}$. Very analogously one gets the $y$-component of the force:

$$\gamma'_u F'_x = \gamma_u F_x\ ; \quad \gamma'_u F'_y = \gamma_u F_y\ . \tag{2.85}$$

For the $z$-component we get:

$$F'_z = \frac{d}{dt'}p'_{\mathrm{r}z} = \frac{\gamma_u}{\gamma'_u}\frac{d}{dt}\left[\gamma_v\left(p_{\mathrm{r}z} - \beta\frac{T_{\mathrm{r}}}{c}\right)\right]\ .$$

Then with (2.53) it follows that:

$$\gamma_u' F_z' = \gamma_v \left( \gamma_u F_z - \beta\, \gamma_u \frac{\mathbf{F} \cdot \mathbf{u}}{c} \right) \,. \tag{2.86}$$

Finally it still remains:

$$\left( \frac{\mathbf{F} \cdot \mathbf{u}}{c} \right)' = \frac{d}{dt'} \frac{T_{\mathrm{r}}'}{c} = \frac{\gamma_u}{\gamma_u'} \frac{d}{dt} \left[ \gamma_v \left( \frac{T_{\mathrm{r}}}{c} - \beta\, p_{\mathrm{rz}} \right) \right] .$$

We read this expression as:

$$\gamma_u' \left( \frac{\mathbf{F} \cdot \mathbf{u}}{c} \right)' = \gamma_v \left[ \gamma_u \left( \frac{\mathbf{F} \cdot \mathbf{u}}{c} \right) - \beta\, \gamma_u\, F_z \right] . \tag{2.87}$$

Let us now define

$$\begin{aligned} K^0 &= \gamma_u \frac{\mathbf{F} \cdot \mathbf{u}}{c} \\ K^1 &= \gamma_u\, F_x \\ K^2 &= \gamma_u\, F_y \\ K^3 &= \gamma_u\, F_z \,. \end{aligned} \tag{2.88}$$

By the use of (2.85) to (2.87) we arrive at:

$$\begin{aligned} K^{0'} &= \gamma_v \left( K^0 - \beta\, K^3 \right) \\ K^{1'} &= K^1 \\ K^{2'} &= K^2 \\ K^{3'} &= \gamma_v \left( K^3 - \beta\, K^0 \right) \,. \end{aligned} \tag{2.89}$$

But these are again the transformation formulae,

$$K^{\mu'} = L_{\mu\,\lambda}\, K^{\lambda} \,, \tag{2.90}$$

of a contravariant four-vector, namely the

$$\textbf{Minkowski force}: \quad K^{\mu} \equiv \left( K^0,\, K^1,\, K^2,\, K^3 \right) \,. \tag{2.91}$$

With (2.38) it immediately follows,

$$\frac{d}{d\tau} = \gamma_u \frac{d}{dt} \qquad (\tau = \text{proper time}) \,, \tag{2.92}$$

the force equation (2.43):

$$K^{\mu} \equiv \frac{d}{d\tau} p^{\mu} = m \frac{d}{d\tau} u^{\mu} \ . \tag{2.93}$$

All the relations of Sect. 2.2.2 are therewith explicitly verified by the discussion of the elastic collision.

There is a last point of the program we have to still deal with. We still have to prove that the integration constant $d$ in (2.77) indeed vanishes as stated in (2.79). In all subsequently derived relations we have to actually replace $T_{\mathrm{r}}$ by $T_{\mathrm{r}} - d$.

Considering the collision process we define a new four-vector

$$\Delta p^{\mu} = \left(\Delta p^0, \Delta \mathbf{p}_{\mathrm{r}}\right) \ , \tag{2.94}$$

where

$$\Delta \mathbf{p}_{\mathrm{r}} = \sum_i \mathbf{p}_{\mathrm{r}}^{(i)} - \sum_f \mathbf{p}_{\mathrm{r}}^{(f)} \tag{2.95}$$

represents the difference of the sum of the relativistic initial momenta ($i$ for *initial*) and the sum of the final momenta ($f$ for *final*). $\Delta p^0$ is the corresponding expression for the time components:

$$\Delta p^0 = \sum_i \left(p^0\right)^{(i)} - \sum_f \left(p^0\right)^{(f)} \ . \tag{2.96}$$

Hence $\Delta p^{\mu}$ is a contravariant four-vector because all the $p^{\mu}$ involved are contravariant four-vectors (2.95), which is the law of conservation of momentum, valid in all inertial systems, takes a very simple form:

$$\Delta \mathbf{p}_{\mathrm{r}} = 0 \ . \tag{2.97}$$

This has the immediate consequence for all inertial systems:

$$\Delta p^0 = 0 \tag{2.98}$$

Because of (2.97), out of $\Delta p^{3'} = \gamma(\Delta p^3 - \beta \, \Delta p^0)$ we have $0 = -\gamma \, \beta \, \Delta p^0$ and therewith (2.98). $\Delta p^{\mu}$ is thus the four-zero vector:

$$0 \stackrel{!}{=} c \, \Delta p^0 = \sum_i (T_{\mathrm{r}})^{(i)} - \sum_f (T_{\mathrm{r}})^{(f)} - \sum_i \left(\varepsilon(0) - m\, c^2\right)^{(i)} + \sum_f \left(\varepsilon(0) - m\, c^2\right)^{(f)} \ .$$

The first two summands cancel each other since the law of conservation of energy, too, has to be valid in all inertial systems to meet the precondition. So what remains is:

$$\sum_f \left(\varepsilon(0) - m\,c^2\right)^{(f)} \stackrel{!}{=} \sum_i \left(\varepsilon(0) - m\,c^2\right)^{(i)} . \tag{2.99}$$

This relation should be fulfilled for arbitrary collisions, e.g. also for those with different particle numbers and particle types (particle transmutations), before and after the collision. But that is possible only if generally it is assumed that

$$\varepsilon(0) = m\,c^2 .$$

This means that the constant $d$ is really zero and the relativistic kinetic energy $T_{\rm r}$ has indeed the form (2.54).

## 2.3 Covariance of Electrodynamics

In the previous section we could recognize that the deviations of the relativistic mechanics from the *'familiar'* Newtonian mechanics become especially drastic when the velocities become comparable to the velocity of light. It therefore certainly comes out as a complete surprise that the

**Maxwell equations of electrodynamics remain valid without any change even for high velocities!**

This is because, they are already form-invariant under Lorentz transformations, which will be shall explicitly demonstrated in this section by reformulating them in terms of four-tensors. In the case of Newtonian mechanics this adaptation was possible only by a redefining of several physical terms like momentum, energy, and force, which solely in the limit $v \ll c$ assume the non-relativistic forms familiar to us from Vol. 1. Such a redefining is not necessary in electrodynamics. In the four-dimensional *'language'* the Maxwell equations are especially simple and symmetric. In particular, they then show the close correlation between electric and magnetic fields which is of particular importance for a deepened understanding of electromagnetic processes. What appears in the one inertial system as magnetic field can manifest itself in another inertial system, at least partly, as electric field and vice versa.

### 2.3.1 Continuity Equation

The experimental observation teaches us that the

**electric charge $q$ is a Lorentz invariant.**

There does not exist the slightest indication that the charge of a particle is dependent on its velocity. However, this does not at all hold for quantities like the

**charge density $\rho$**

or the

**current density $\mathbf{j} = \rho\,\mathbf{v}$**

The reason is evident and is, in the final analysis, related to the length contraction.

Let $\Sigma_0$ be a (co-moving) inertial system in which the considered charge is at rest:

$$dq = \rho_0\, dV_0 \ .$$

Hence, $\rho_0$ is the co-moving charge density.

Let $\Sigma$ be another inertial system moving relative to $\Sigma_0$ with the velocity $\mathbf{v}$ parallel to the $z$-axis. Since the amount of charge cannot have changed in the given volume element, we can set:

$$dq = \rho\, dV \ .$$

For the volume element $dV$ the length contraction (1.28) leads to:

$$dV = dx\, dy\, dz = dx_0\, dy_0\, dz_0 \sqrt{1 - v^2/c^2} = \frac{1}{\gamma} dV_0 \ .$$

From $\rho\, dV = \rho_0\, dV_0$ it then follows for the charge density seen from $\Sigma$:

$$\rho = \rho_0\, \gamma \ . \tag{2.100}$$

According to its meaning as *rest-charge density* $\rho_0$ must be considered as Lorentz invariant. The charge density $\rho$ produces in $\Sigma$ a current density $\mathbf{j}$:

$$\mathbf{j} = \gamma\, \rho_0\, \mathbf{v} \ . \tag{2.101}$$

One recognizes from Eqs. (2.100) and (2.101) a contravariant four-vector, the so-called

**four-current density**

$$j^\mu = \left(c\rho, j_x, j_y, j_z\right) \equiv (c\rho, \mathbf{j}) = \gamma\, \rho_0 (c, \mathbf{v}) = \rho_0\, u^\mu \ . \tag{2.102}$$

That it is indeed a contravariant four-vector follows from the fact that $\rho_0$ is a four-scalar. $j^\mu$ thus transforms as the world-velocity $u^\mu$, about which we already know that it is such a contravariant four-vector. Nevertheless, we prove it explicitly.

In the rest system $\Sigma_0$ of the charge we have:

$$\mathbf{j}_0 = \rho_0\,\mathbf{v}_0 = \mathbf{0}\ ; \quad j_0^0 = c\,\rho_0 \quad \longrightarrow \quad j_0^\mu = (c\rho_0, 0, 0, 0)\ .$$

The Lorentz transformation (1.16) then yields for the components of the four-current density in $\Sigma$:

$$\begin{aligned}
j^0 &= c\rho = \gamma\, j_0^0 - \beta\gamma\, j_0^3 = c\,\gamma\rho_0\ ,\\
j^1 &= j_x = j_{0x} = 0\ ,\\
j^2 &= j_y = j_{0y} = 0\ ,\\
j^3 &= j_z = -\beta\gamma\, j_0^0 + \gamma\, j_0^3 = -\gamma\,\rho_0\,v = -\rho v\ .
\end{aligned}$$

This is obviously the correct result when one bears in mind that $\mathbf{v} = (0, 0, -v)$ is the velocity of the charge in $\Sigma$.

We now consider the **continuity equation** (see (2.10), Vol. 3):

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\mathbf{j} = 0\ .$$

We recognize on the left-hand side the divergence of the four-vector $j^\mu$ as we see by applying (2.32):

$$\partial_\mu j^\mu = \frac{1}{c}\,\frac{\partial}{\partial t} j^0 + \mathrm{div}\mathbf{j} = \mathrm{div}\mathbf{j} + \frac{\partial}{\partial t}\rho\ .$$

The continuity equation can therefore be written in the very compact form:

$$\partial_\mu j^\mu = 0\ . \tag{2.103}$$

Both sides of the equation are four-scalars. The continuity equation is thus Lorentz invariant.

### *2.3.2 Electromagnetic Potentials*

We now discuss the wave equations of the electromagnetic potentials,

$$\varphi(\mathbf{r}, t):\ \text{scalar potential}\ ; \quad \mathbf{A}(\mathbf{r}, t):\ \text{vector potential}\ ,$$

and we repeat, for this purpose, some considerations from Sect. 4.1.3 in Vol. 3. We want to use here also the international system of units (SI) (Sect. 2.1.2, Vol. 3), although the Gaussian system of units is actually better suited to the Special Theory of Relativity. We know that the **Maxwell equations** can be grouped into two homogeneous and two inhomogeneous differential equations:

$$\begin{aligned} \text{homogeneous}: \quad & \text{div}\mathbf{B} = \mathbf{0}\,, \\ & \text{curl}\mathbf{E} + \dot{\mathbf{B}} = \mathbf{0}\,, \\ \text{inhomogeneous}: \quad & \text{div}\mathbf{D} = \rho\,, \\ & \text{curl}\mathbf{H} - \dot{\mathbf{D}} = \mathbf{j}\,. \end{aligned}$$

We restrict our considerations to the vacuum for which we have to use:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}\,; \quad \mathbf{B} = \mu_0 \mathbf{H}$$

The vector potential is defined by the ansatz ((3.34), Vol. 3),

$$\mathbf{B} = \text{curl}\mathbf{A}\,.$$

The second homogeneous Maxwell equation then reads

$$\text{curl}\left(\mathbf{E} + \dot{\mathbf{A}}\right) = \mathbf{0}\,,$$

which leads to the following ansatz for the electric field **E**:

$$\mathbf{E} = -\text{grad}\varphi - \dot{\mathbf{A}} \qquad ((4.21), \text{Vol. 3})\,.$$

$\varphi$ and **A** are not uniquely determined by the above definition equations. One still has the free choice of a function $\chi(\mathbf{r}, t)$ provided it is chosen so that

$$\varphi \to \varphi - \dot{\chi}\,; \quad \mathbf{A} \to \mathbf{A} + \text{grad}\chi$$

is guaranteed ((4.22) and (4.23), Vol. 3). Because of curlgrad$\chi = 0$ such a **gauge transformation** does not change the fields **E** and **B**. One therefore can fix it under practicality aspects.

By the introduction of the electromagnetic potentials $\varphi$ and **A** the homogeneous Maxwell equations are automatically fulfilled, while the two inhomogeneous equations become now differential equations of second order for the potentials $\varphi$ and **A**. These equations, however, assume an especially symmetric shape if one chooses the gauge function $\chi(\mathbf{r}, t)$ such that the

**Lorenz condition**

$$\text{div}\mathbf{A} + \frac{1}{c^2}\dot{\varphi} = \text{div}\mathbf{A} + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\varphi\right) = 0 \tag{2.104}$$

is fulfilled (see (4.37), Vol. 3). With $c = (\mu_0 \varepsilon_0)^{-1/2}$ the potentials $\varphi$ and $\mathbf{A}$ are solutions of the following

**wave equations**

$$\Box\,\mathbf{A} \equiv \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = -\mu_0\,\mathbf{j}\,, \tag{2.105}$$

$$\Box\left(\frac{1}{c}\varphi\right) = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{1}{c}\varphi\right) = -\frac{1}{c}\frac{\rho}{\varepsilon_0} = -\mu_0(c\,\rho) \tag{2.106}$$

(see (4.38) and (4.39), Vol. 3). On the right-hand sides of these wave equations we recognize the space and time components of the four-current density $j^\mu$. Since the d'Alembert operator $\Box$ (2.33) is a scalar operator, Eqs. (2.105) and (2.106) make it appear reasonable to introduce a further four-vector, the

**four-potential**

$$A^\mu \equiv \left(\frac{1}{c}\varphi, A_x, A_y, A_z\right) \equiv \left(\frac{1}{c}\varphi, \mathbf{A}\right)\,. \tag{2.107}$$

The wave equations for $\varphi$ and $\mathbf{A}$ can therewith be combined as

**four-wave equation**

$$\Box A^\mu = -\mu_0 j^\mu\,, \tag{2.108}$$

which is covariant since both sides are four-tensors of the same, namely, the first rank.

The Lorenz condition (2.104) can finally still be written as four-divergence (2.32) of the potential $A^\mu$. The relation

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial}{\partial t}A^0 + \mathrm{div}\mathbf{A}$$

is obviously identical to the left-hand side of (2.104). The

**Lorenz condition**

$$\partial_\mu A^\mu \equiv 0 \tag{2.109}$$

is as world-scalar Lorentz invariant.

### *2.3.3 Field-Strength Tensors*

The field strengths $\mathbf{E}$ and $\mathbf{B}$ can not be written in the relativistic electrodynamics as four-vectors. Instead of that, we have to introduce for the fields, a four-tensor of second rank which incorporates likewise the fields **E** and **B**. Starting point is again the connection between fields and potentials:

$$\mathbf{B} = \text{curl}\mathbf{A} \; ; \quad \mathbf{E} = -\text{grad}\varphi - \dot{\mathbf{A}} \; .$$

We already introduced the four-gradient in Sect. 2.1.3:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = (\partial_0, \partial_1, \partial_2, \partial_3) \; , \tag{2.110}$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) = \left(\partial^0, \partial^1, \partial^2, \partial^3\right) \; . \tag{2.111}$$

It holds obviously:

$$\partial_0 = \partial^0 \; ; \quad \partial_{1,2,3} = -\partial^{1,2,3} \; . \tag{2.112}$$

This we use to reformulate at first the **B**-field:

$$B_x = \frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y = \partial_2 A^3 - \partial_3 A^2 = -\left(\partial^2 A^3 - \partial^3 A^2\right) \; .$$

Analogously, it follows for the two other Cartesian components:

$$B_y = -\left(\partial^3 A^1 - \partial^1 A^3\right) \; ; \quad B_z = -\left(\partial^1 A^2 - \partial^2 A^1\right) \; .$$

The **E**-field can be written quite similarly :

$$E_x = -\frac{\partial}{\partial x}\varphi - \frac{\partial}{\partial t}A_x = -c\left[\frac{\partial}{\partial x}\left(\frac{1}{c}\varphi\right) + \frac{1}{c}\frac{\partial}{\partial t}A_x\right]$$
$$= c\left(\partial^1 A^0 - \partial^0 A^1\right) \; .$$

Corresponding expressions are valid for $E_y$ and $E_z$:

$$E_y = c\left(\partial^2 A^0 - \partial^0 A^2\right) \; ; \quad E_z = c\left(\partial^3 A^0 - \partial^0 A^3\right) \; .$$

We introduce by

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.113}$$

a new **four-tensor of second rank**. As a tensor of two contravariant four-vectors it is also **contravariant** and is obviously antisymmetric:

$$F^{\mu\nu} = -F^{\nu\mu} \; . \tag{2.114}$$

One can regard this tensor as four-dimensional generalization of the curl of the vector $A^{\mu}$:

**field-strength tensor**

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{2.115}$$

The electromagnetic field is described in the Minkowski space not anymore by **two** fields but rather by **one** tensor of second rank. In the next section, we will use the field-strength tensor for the covariant formulation of the Maxwell equations.

The *covariant field-strength tensor* comes out easily by use of the general *'conversion-prescription'* (2.26) presented in Sect. 2.1.2:

$$F_{\mu\nu} = \mu_{\mu\alpha}\mu_{\nu\beta}F^{\alpha\beta} . \tag{2.116}$$

Since the metric tensor $\mu_{\alpha\beta}$ is diagonal in the Special Theory of Relativity (2.19), it simply follows:

$$F_{0\nu} = -F^{0\nu} ; \quad F_{\nu 0} = -F^{\nu 0} ; \quad F_{\mu\nu} = F^{\mu\nu} ; \quad \mu\nu \in \{1, 2, 3\} . \tag{2.117}$$

So we have to replace in (2.115) only **E** by −**E** in order to come from $F^{\mu\nu}$ to $F_{\mu\nu}$. We read off from (2.115) an important

**invariant of the electromagnetic field**

$$F_{\mu\nu}F^{\mu\nu} = 2\left(\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2\right) , \tag{2.118}$$

which as a four-scalar remains uninfluenced by Lorentz transformations. It is obviously never possible to transform a **pure B**-field into a **pure E**-field, or vice versa, since the two terms in (2.118) carry different signs. Later we will once more come back to this fact.

### *2.3.4 Maxwell Equations*

We now want to derive with the aid of the field-strength tensor (2.115) the Maxwell equations in explicit covariant form. We start with the **inhomogeneous** equations which can be written with $c = (\varepsilon_0\mu_0)^{-(1/2)}$ as follows:

$$\text{div}\left(\frac{1}{c}\mathbf{E}\right) = \mu_0\, c\rho = \mu_0 j^0 , \tag{2.119}$$

$$\text{curl}\mathbf{B} - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{E}\right) = \mu_0\mathbf{j} . \tag{2.120}$$

On the right-hand side of these equations we identify the components of the four-current $j^{\mu}$ (2.102). The left-hand sides should therefore also be components of a four vector, if, as initially stated, the system of the Maxwell equations is indeed covariant. Let us try to express the left-hand sides by the field-strength tensor:

$\boxed{\mu = 0}$

$$\begin{aligned}\mu_0 j^{0} &= \operatorname{div}\left(\tfrac{1}{c}\mathbf{E}\right) = \tfrac{1}{c}\left(\tfrac{\partial}{\partial x}E_x + \tfrac{\partial}{\partial y}E_y + \tfrac{\partial}{\partial z}E_z\right)\\ &= \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \partial_\alpha F^{\alpha 0}\ .\end{aligned}$$

$\boxed{\mu = 1}$

$$\begin{aligned}\mu_0 j^{1} &= \mu_0 j_x = \tfrac{\partial}{\partial y}B_z - \tfrac{\partial}{\partial z}B_y + \tfrac{1}{c}\tfrac{\partial}{\partial t}\left(-\tfrac{1}{c}E_x\right)\\ &= \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \partial_\alpha F^{\alpha 1}\ .\end{aligned}$$

$\boxed{\mu = 2}$

$$\begin{aligned}\mu_0 j^{2} &= \mu_0 j_y = \tfrac{\partial}{\partial z}B_x - \tfrac{\partial}{\partial x}B_z + \tfrac{1}{c}\tfrac{\partial}{\partial t}\left(-\tfrac{1}{c}E_y\right)\\ &= \partial_3 F^{32} + \partial_1 F^{12} + \partial_0 F^{02} = \partial_\alpha F^{\alpha 2}\ .\end{aligned}$$

$\boxed{\mu = 3}$

$$\begin{aligned}\mu_0 j^{3} &= \mu_0 j_z = \tfrac{\partial}{\partial x}B_y - \tfrac{\partial}{\partial y}B_x + \tfrac{1}{c}\tfrac{\partial}{\partial t}\left(-\tfrac{1}{c}E_z\right)\\ &= \partial_1 F^{13} + \partial_2 F^{23} + \partial_0 F^{03} = \partial_\alpha F^{\alpha 3}\ .\end{aligned}$$

These relations can be summed up to a compact expression:

**inhomogeneous Maxwell equations**

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^{\beta}\ ; \quad \beta = 0, 1, 2, 3\ . \tag{2.121}$$

On the left we have a *contracted* third-rank tensor, consequently a four-vector like the one on the right-hand side. Covariance is therewith guaranteed. The inhomogeneous Maxwell equations in this form are valid in all inertial systems.

We now consider the **homogeneous** Maxwell equations:

$$\operatorname{div}\mathbf{B} = 0\ , \tag{2.122}$$

$$\operatorname{curl}\left(\frac{1}{c}\mathbf{E}\right) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} = 0\ . \tag{2.123}$$

For (2.122) we can also write with (2.112) and (2.114):

$$0 = \mathrm{div}\mathbf{B} = \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y + \frac{\partial}{\partial z}B_z = \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21}$$
$$= \left(\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12}\right) .$$

The three components of the vector-equation (2.123) can be reformulated with (2.112) in the following manner:

$$0 = \left(\mathrm{curl}\frac{1}{c}\mathbf{E}\right)_x + \frac{1}{c}\frac{\partial}{\partial t}B_x = \frac{\partial}{\partial y}\left(\frac{1}{c}E_z\right) - \frac{\partial}{\partial z}\left(\frac{1}{c}E_y\right) + \frac{1}{c}\frac{\partial}{\partial t}B_x$$
$$= \partial_2 F^{30} + \partial_3 F^{02} - \partial_0 F^{23} = -\left(\partial^2 F^{30} + \partial^3 F^{02} + \partial^0 F^{23}\right) ,$$
$$0 = \left(\mathrm{curl}\frac{1}{c}\mathbf{E}\right)_y + \frac{1}{c}\frac{\partial}{\partial t}B_y = \frac{\partial}{\partial z}\left(\frac{1}{c}E_x\right) - \frac{\partial}{\partial x}\left(\frac{1}{c}E_z\right) + \frac{1}{c}\frac{\partial}{\partial t}B_y$$
$$= \partial_3 F^{10} + \partial_1 F^{03} - \partial_0 F^{31} = -\left(\partial^3 F^{10} + \partial^1 F^{03} + \partial^0 F^{31}\right) ,$$
$$0 = \left(\mathrm{curl}\frac{1}{c}\mathbf{E}\right)_z + \frac{1}{c}\frac{\partial}{\partial t}B_z = \frac{\partial}{\partial x}\left(\frac{1}{c}E_y\right) - \frac{\partial}{\partial y}\left(\frac{1}{c}E_x\right) + \frac{1}{c}\frac{\partial}{\partial t}B_z$$
$$= \partial_1 F^{20} + \partial_2 F^{01} - \partial_0 F^{12} = -\left(\partial^1 F^{20} + \partial^2 F^{01} + \partial^0 F^{12}\right) .$$

These equations, too, can be brought into a compact form:

**homogeneous Maxwell equations**

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 , \quad \alpha, \beta, \gamma \text{ arbitrary from } (0, 1, 2, 3) . \tag{2.124}$$

All additive terms in this expression, which differ from one another by a cyclic interchange of the indexes $\alpha, \beta, \gamma$, are four-tensors of the same rank. If two indexes in (2.124) are equal, then the left-hand side is identical to zero. So it follows, for instance, for $\alpha = \beta$ (2.114):

$$\partial^\alpha F^{\alpha\gamma} + \partial^\alpha F^{\gamma\alpha} + \partial^\gamma F^{\alpha\alpha} = \partial^\alpha \left(F^{\alpha\gamma} - F^{\alpha\gamma}\right) = 0 .$$

Hence, only the combinations $(0, 1, 3)$, $(0, 1, 2)$, $(1, 2, 3)$ are of interest. But these are just the above discussed four homogeneous Maxwell equations.

Equations (2.121) and (2.124) demonstrate that the system of Maxwell equations can be expressed by four-tensors in a very compact and symmetric form, where the covariance with respect to Lorentz transformations becomes immediately evident.

A still more compact representation of the homogeneous Maxwell equations than that in (2.124) is achieved by introducing the so-called **dual field-strength tensor**:

$$\overline{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} . \tag{2.125}$$

Here we use the totally antisymmetric fourth-rank tensor

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1\,, & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3)\,, \\ -1\,, & \text{if it is an odd permutation}\,, \\ 0\,, & \text{if two or more indexes are equal}\,, \end{cases} \tag{2.126}$$

The elements $F_{\mu\nu}$ of the covariant field-strength tensor are related by (2.117) to those of the contravariant tensor (2.115). At first, one reads off immediately from the definition (2.125):

$$\overline{F}^{\mu\nu} = -\overline{F}^{\nu\mu} \tag{2.127}$$

Hence the diagonal elements are all zero. Let us calculate as an example under consideration of (2.117):

$$\begin{aligned} \overline{F}^{12} &= \frac{1}{2}\varepsilon^{12\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{1230}F_{30} + \varepsilon^{1203}F_{03}\right) \\ &= \frac{1}{2}\left(-F_{30} + F_{03}\right) = F^{30} = \frac{1}{c}E_z\,. \end{aligned}$$

Fully analogously one finds (see Exercise 2.5.10, only the elements with $\mu < \nu$ need to be calculated):

$$\overline{F}^{13} = F^{02}\,; \quad \overline{F}^{01} = F^{23}\,; \quad \overline{F}^{23} = F^{10}\,; \quad \overline{F}^{02} = F^{31}\,; \quad \overline{F}^{03} = F^{12}\,.$$

According to that one obtains the components of the dual field-strength tensor $\overline{F}^{\mu\nu}$ from those of the covariant tensor $F_{\mu\nu}$ by the replacement:

$$\mathbf{B} \longleftrightarrow -\frac{1}{c}\mathbf{E}\,. \tag{2.128}$$

This yields with (2.115):

$$\overline{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{1}{c}E_z & -\frac{1}{c}E_y \\ B_y & -\frac{1}{c}E_z & 0 & \frac{1}{c}E_x \\ B_z & \frac{1}{c}E_y & -\frac{1}{c}E_x & 0 \end{pmatrix}\,. \tag{2.129}$$

It is now easy to check the following relations:

$$\begin{aligned} \partial_\alpha \overline{F}^{\alpha 0} &= \partial_0 \overline{F}^{00} + \partial_1 \overline{F}^{10} + \partial_2 \overline{F}^{20} + \partial_3 \overline{F}^{30} \\ &= \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12}\,, \\ \partial_\alpha \overline{F}^{\alpha 1} &= \partial_0 F^{23} + \partial_2 F^{03} + \partial_3 F^{20} = \partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02}\,, \end{aligned}$$

$$\partial_\alpha \overline{F}^{\alpha 2} = \partial^0 F^{31} + \partial^1 F^{03} + \partial^3 F^{10} ,$$
$$\partial_\alpha \overline{F}^{\alpha 3} = \partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} .$$

We thus can also write in lieu of (2.124):

**homogeneous Maxwell equations**

$$\partial_\alpha \overline{F}^{\alpha\beta} = 0 ; \quad \beta = 0, 1, 2, 3 . \tag{2.130}$$

Eventually, one still recognizes with (2.115) and (2.129) a further

**invariant of the electromagnetic field**

$$F_{\alpha\beta}\overline{F}^{\alpha\beta} = -\frac{4}{c}\,\mathbf{E}\cdot\mathbf{B} . \tag{2.131}$$

A four-tensor of zeroth rank appears on the left side, i.e., a world-scalar. The scalar product of the electric and the magnetic field $\mathbf{E}\cdot\mathbf{B}$ therefore does not change under a Lorentz transformation being therewith the same in all inertial systems.

### *2.3.5 Transformation of the Electromagnetic Fields*

With the relations derived in the last section it is now easy to calculate how electric and magnetic fields behave in detail under a Lorentz transformation.

We got to know in Sect. 2.1 the transformation behavior of a contravariant second-rank tensor. Equations (2.8) and (2.9),

$$(F^{\mu\nu})' = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} F^{\alpha\beta} = L_{\mu\alpha}L_{\nu\beta}F^{\alpha\beta} ,$$

lead with (1.16) to the transformed field-strength tensor. Because of $F^{\alpha\beta} = -F^{\beta\alpha}$ it also follows immediately

$$(F^{\mu\nu})' = -(F^{\nu\mu})' .$$

This means in particular that the diagonal elements of the transformed tensor vanish. There are therefore only six elements to be explicitly calculated:

$$\left(F^{01}\right)' = L_{11}\left(L_{00}F^{01} + L_{03}F^{31}\right) = \gamma\left(-\frac{1}{c}E_x + \beta B_y\right) \stackrel{!}{=} -\frac{1}{c}E'_x ,$$
$$\left(F^{02}\right)' = L_{22}\left(L_{00}F^{02} + L_{03}F^{32}\right) = \gamma\left(-\frac{1}{c}E_y - \beta B_x\right) \stackrel{!}{=} -\frac{1}{c}E'_y ,$$
$$\left(F^{03}\right)' = L_{00}\left(L_{30}F^{00} + L_{33}F^{03}\right) + L_{03}\left(L_{30}F^{30} + L_{33}F^{33}\right)$$

$$= \gamma^2 \left(1-\beta^2\right)\left(-\frac{1}{c}E_z\right) = -\frac{1}{c}E_z \stackrel{!}{=} -\frac{1}{c}E_z' \,,$$

$$\left(F^{12}\right)' = L_{11}L_{22}F^{12} = -B_z \stackrel{!}{=} -B_z' \,,$$

$$\left(F^{13}\right)' = L_{11}\left(L_{30}F^{10} + L_{33}F^{13}\right) = \gamma\left(B_y - \frac{\beta}{c}E_x\right) \stackrel{!}{=} B_y' \,,$$

$$\left(F^{23}\right)' = L_{22}\left(L_{30}F^{20} + L_{33}F^{23}\right) = \gamma\left(-\frac{\beta}{c}E_y - B_x\right) \stackrel{!}{=} -B_x' \,.$$

The components of the magnetic field transform therewith according to the following formulae:

$$B_x' = \gamma\left(B_x + \frac{\beta}{c}E_y\right) \,, \tag{2.132}$$

$$B_y' = \gamma\left(B_y - \frac{\beta}{c}E_x\right) \,, \tag{2.133}$$

$$B_z' = B_z \,. \tag{2.134}$$

For the components of the electric field we have:

$$E_x' = \gamma\left(E_x - \beta\, c\, B_y\right) \,, \tag{2.135}$$

$$E_y' = \gamma\left(E_y + \beta\, c\, B_x\right) \,, \tag{2.136}$$

$$E_z' = E_z \,. \tag{2.137}$$

These formulae demonstrate the close entanglement of electric and magnetic fields. That what appears in one system as pure **E**- or **B**-field is in an other inertial system a mixture of both. Under all Lorentz transformations, however, the

**invariants of the electromagnetic field**

$$\left(\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2\right) \quad \text{and} \quad \mathbf{E}\cdot\mathbf{B}$$

remain unchanged. From that it follows, as we have already stated earlier, that it is impossible to transform a pure **B**-field into a pure **E**-field and vice versa.

For certain purposes it appears reasonable to resolve the fields into components parallel and perpendicular to the relative velocity **v** of the two inertial systems $\Sigma$ and $\Sigma'$. The parallel-component of the electric field is of course identical to the $z$-component:

$$\mathbf{E}_{\parallel}' = E_z'\mathbf{e}_z = E_z\mathbf{e}_z = \mathbf{E}_{\parallel} \,. \tag{2.138}$$

The orthogonal component is determined slightly more complicatedly:

$$\mathbf{E}'_\perp = E'_x\mathbf{e}_x + E'_y\mathbf{e}_y = \gamma\left[E_x\mathbf{e}_x + E_y\mathbf{e}_y + \beta\, c\left(B_x\mathbf{e}_y - B_y\mathbf{e}_x\right)\right] \;.$$

With

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c} \equiv \left(0, 0, \frac{v}{c}\right) \tag{2.139}$$

the last summand can be written as a vector product:

$$\mathbf{E}'_\perp = \gamma\left[\mathbf{E}_\perp + c(\boldsymbol{\beta}\times\mathbf{B})\right] \;. \tag{2.140}$$

Analogously one finds for the magnetic induction:

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel \;;\quad \mathbf{B}'_\perp = \gamma\left[\mathbf{B}_\perp - \frac{1}{c}(\boldsymbol{\beta}\times\mathbf{E})\right] \;. \tag{2.141}$$

These results for the electromagnetic fields can be combined as follows:

$$\mathbf{E}' = \gamma\left[\mathbf{E} + c(\boldsymbol{\beta}\times\mathbf{B})\right] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{E}) \;, \tag{2.142}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c}(\boldsymbol{\beta}\times\mathbf{E})\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{B}) \;. \tag{2.143}$$

In this form the transformation formulae are valid for arbitrary velocities $\mathbf{v}$. $\mathbf{v}$ need not necessarily be directed parallel to the $z$-axis. Let us check it:

$$\begin{aligned}
\mathbf{E}'_\parallel &\equiv \frac{\boldsymbol{\beta}}{\beta}\left(\mathbf{E}'\cdot\frac{\boldsymbol{\beta}}{\beta}\right) = \frac{\boldsymbol{\beta}}{\beta^2}\left(\gamma\;\mathbf{E}\cdot\boldsymbol{\beta} - \frac{\gamma^2}{\gamma+1}\beta^2(\boldsymbol{\beta}\cdot\mathbf{E})\right)\\
&= \mathbf{E}_\parallel\left(\gamma - \frac{\gamma^2}{\gamma+1}\beta^2\right) = \mathbf{E}_\parallel\frac{\gamma^2+\gamma-\gamma^2\beta^2}{\gamma+1} = \mathbf{E}_\parallel \;,\\
\mathbf{E}'_\perp &= \mathbf{E}' - \frac{\boldsymbol{\beta}}{\beta}\left(\mathbf{E}'\cdot\frac{\boldsymbol{\beta}}{\beta}\right) = \mathbf{E}' - \mathbf{E}_\parallel\\
&= \gamma\left[\mathbf{E}_\perp + c(\boldsymbol{\beta}\times\mathbf{B})\right] + \mathbf{E}_\parallel\left(\gamma - \frac{\gamma^2}{\gamma+1}\beta^2 - 1\right) = \gamma\left[\mathbf{E}_\perp + c(\boldsymbol{\beta}\times\mathbf{B})\right] \;.
\end{aligned}$$

These are for the **E**-field the correct expressions (2.138) and (2.140). The procedure for the **B**-field succeeds in the same manner.

We discuss as an **example of application use** the

**fields of a moving point charge $q$.**

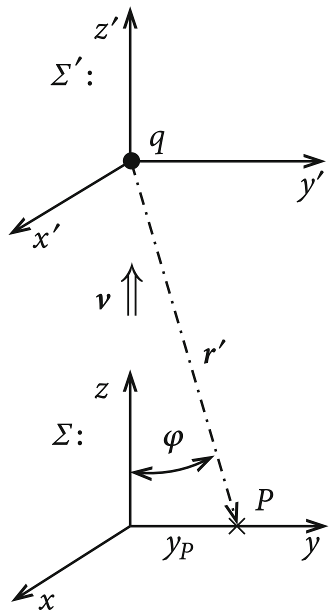



**Fig. 2.2** For the calculation of the fields of a moving point charge in two inertial systems moving relative to one another

We assume that the charge $q$ is located at the origin of $\Sigma'$. Let $\Sigma'$ be the rest system of the charge. On the other hand, let $\Sigma$ be the rest system of an observer $P$, where we can assume, without loss of generality, that his position lies on the $y$-axis in $\Sigma$. We assume that $\Sigma'$ moves relative to $\Sigma$ with the constant velocity $\mathbf{v}$ in $z$-direction, where at $t = t' = 0$ both the systems of coordinates coincide (Fig. 2.2). The position of $P$ in $\Sigma$ is given by

$$\mathbf{r}_P = (0, y_P, 0)$$

and in $\Sigma'$ by

$$\mathbf{r}' = \left(0, y_P, -v\,t'\right) \ . \qquad (2.144)$$

The distance of the observer from the point charge

$$r' = \sqrt{y_P^2 + v^2 t'^2}$$

is, because of

$$t' = \gamma\left(t - \frac{v}{c^2} z_P\right) = \gamma\, t \ ,$$

in Σ-coordinates:

$$r' = \sqrt{y_P^2 + v^2\gamma^2 t^2} \; . \tag{2.145}$$

Which fields are now seen by the observer $P$ when the point charge $q$ moves with $v = \text{const}$ along the $z$-axis?

The fields in $\Sigma'$ are those of a charge **at rest**, therefore known from non-relativistic electrodynamics (see Vol. 3):

$$\mathbf{B}' \equiv 0 \; , \tag{2.146}$$

$$\mathbf{E}'(t') = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3} = \frac{q}{4\pi\varepsilon_0 r'^3}\left(0, y_P, -v\,t'\right) \; . \tag{2.147}$$

The **E**-field is time-dependent since the position of the observer $P$ in $\Sigma'$ changes in the course of time, i.e., the field is measured for different times $t'$ at different places $\mathbf{r}'$ in $\Sigma'$ (Fig. 2.3).

For the calculation of the **E**-field in Σ we use the transformation formulae (2.135) to (2.137), where it is to bear in mind that Σ moves relative to $\Sigma'$ with the velocity $-\mathbf{v}$:

$$E_x = \gamma\left(E'_x + \beta\,c\,B'_y\right) = 0 \; , \tag{2.148}$$

$$E_y = \gamma\left(E'_y - \beta\,c\,B'_x\right) = \gamma\,E'_y = \frac{q}{4\pi\varepsilon_0}\frac{\gamma\,y_P}{\left(y_P^2 + \gamma^2 v^2 t^2\right)^{3/2}} \; , \tag{2.149}$$

$$E_z = E'_z = \frac{q}{4\pi\varepsilon_0}\frac{-\gamma\,v\,t}{\left(y_P^2 + \gamma^2 v^2 t^2\right)^{3/2}} \; . \tag{2.150}$$

The **transverse** component $E_y$ is obviously remarkably different from zero only in the time interval

$$-t_{1/2} \le t \le +t_{1/2} \; ,$$

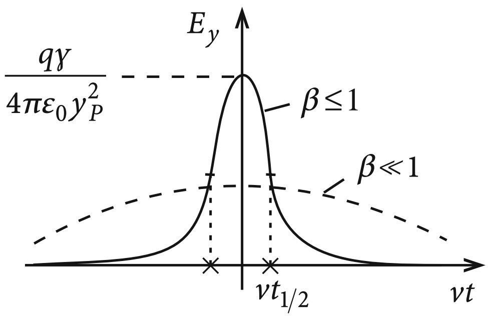



**Fig. 2.3** Time-dependence of the transverse component $E_y$ of the electric field of a point charge, which moves relative to the observer in $z$-direction, seen from the rest system of the observer

where the **half width** $t_{1/2}$ is defined by

$$\frac{E_y\left(t = t_{1/2}\right)}{E_y(t=0)} \stackrel{!}{=} \frac{1}{2}\,.$$

It is thus fixed by

$$\frac{1}{2} \stackrel{!}{=} \frac{y_P^3}{\left(y_P^2 + \gamma^2 v^2 t_{1/2}^2\right)^{3/2}}$$

as

$$t_{1/2} = \frac{y_P}{\gamma\, v}\left(2^{2/3} - 1\right)^{1/2}\,.$$

The maximum of the $E_y$-peak appears at $t = 0$, hence at the moment of the minimum spatial distance between charge $q$ and $P$. In the relativistic region $v \lesssim c$ the peak becomes very sharp because $t_{1/2}$ is then very small.
The **longitudinal** component $E_z$ of the electric field (2.150) changes its sign (Fig. 2.4) at $t = 0$ and has extrema at

$$\pm t_0 = \frac{y_P}{\gamma\, v}\frac{1}{\sqrt{2}}\,,$$

This is easily proved by putting the first time-derivative equal to zero. The extreme values,

$$E_z\left(t = \pm t_0\right) = \mp\frac{q}{6\sqrt{3}\pi\,\varepsilon_0 y_P^2}\,,$$

are thereby $v$-independent.

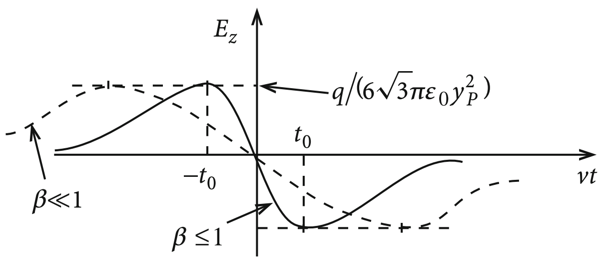



**Fig. 2.4** Time- dependence of the longitudinal component $E_z$ of the electric field of a point charge, which moves relative to the observer in $z$-direction, seen from the rest system of the observer

How does the spatial distribution of the electric field **E** relative to the point charge look like in the observer-system $\Sigma$ at a fixed time $t$? Figure 2.2 illustrates

$$y_P = r \sin\varphi \ ; \quad v\,t = r\cos\varphi \ ,$$

where $r$ is the distance between charge and observer in $\Sigma$. It follows with

$$\begin{aligned}\left(y_P^2 + \gamma^2 v^2 t^2\right)^{3/2} &= r^3\left(\sin^2\varphi + \gamma^2\cos^2\varphi\right)^{3/2}\\ &= r^3\gamma^3\left(1-\beta^2\sin^2\varphi\right)^{3/2}\end{aligned}$$

for the field distribution in $\Sigma$ around the point charge $q$:

$$\mathbf{E} = \frac{q\,\mathbf{r}}{4\pi\varepsilon_0 r^3\gamma^2\left(1-\beta^2\sin^2\varphi\right)^{3/2}} \ . \tag{2.151}$$

**r** is the space vector from the point charge to the observer. The **E**-field is radial as for a resting charge, but for $\beta \neq 0$ it is no longer isotropic. **In** the direction of motion ($\varphi = 0$ or $\pi$) we have:

$$\frac{E(\beta)}{E(0)} = \frac{1}{\gamma^2} \ ,$$

and, in contrast, **perpendicular** to the direction of motion ($\varphi = \pi/2$):

$$\frac{E(\beta)}{E(0)} = \gamma \ .$$

We still have to discuss the **B**-field of the moving point charge. With the transformation formulae (2.132) to (2.134) we find for the Cartesian components in $\Sigma$:

$$B_x = \gamma\left(B_x' - \frac{\beta}{c}E_y'\right) = -\gamma\frac{\beta}{c}E_y' = -\frac{\beta}{c}E_y \ , \tag{2.152}$$

$$B_y = \gamma\left(B_y' + \frac{\beta}{c}E_x'\right) = 0 \ , \tag{2.153}$$

$$B_z = B_z' = 0 \ . \tag{2.154}$$

For this evaluation we have exploited (2.146) and (2.147). We see that the moving point charge generates a magnetic induction in $x$-direction. Its time-dependence corresponds to that of $E_y$, which we have discussed in detail. A bit more generally one finds for the magnetic induction according to (2.143) with $\mathbf{B}' \equiv 0$:

$$\mathbf{B} = \frac{\gamma}{c}\left(\boldsymbol{\beta}\times\mathbf{E}'\right) = \frac{\mu_0}{4\pi}\frac{q\,\gamma}{r'^3}\left(\mathbf{v}\times\mathbf{r}'\right) \ . \tag{2.155}$$

For $\gamma = 1$ this is the **Biot and Savart law** (see (3.23), Vol. 3) for the magnetic field produced by a moving charge. In our special case ($\mathbf{v} = v\,\mathbf{e}_z$) we have

$$\mathbf{v} \times \mathbf{r}' = -v\, y_P\, \mathbf{e}_x \ . \tag{2.156}$$

## 2.3.6 Lorentz Force

Let us finally add to our four-dimensional formalism still the Lorentz force ((4.40), Vol. 3)

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

At first we interpret $\mathbf{F}$ as the time derivative of the space components ($\mathbf{p}_\mathrm{r}$, (2.58)) of the four-momentum (2.57):

$$p^\mu = \left(\frac{T_\mathrm{r}}{c}, \mathbf{p}_\mathrm{r}\right) = m\, u^\mu = m\gamma\,(c, \mathbf{v}) \ .$$

However, $(d/dt)\mathbf{p}_\mathrm{r}$ are by themselves, not yet, the space components of a four-vector since the time $t$ does not transform accordingly. We therefore introduce again the proper time $\tau$ $(t = \gamma\,\tau)$ and try to generalize the Lorentz force in such a manner that, as in mechanics, a covariant force law comes out:

$$K^\mu \equiv \frac{d}{d\tau} p^\mu \equiv m \frac{d}{d\tau} u^\mu \tag{2.157}$$

On the left-hand side there should of course appear the
**Minkowski force (2.51)**

$$K^\mu \equiv \gamma \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}, F_x, F_y, F_z\right) \equiv \left(K^0, \mathbf{K}\right) \tag{2.158}$$

which we discussed in detail in Sect. 2.2.2 for the case of classical mechanics. With (2.92) at first it follows for the space components

$$\mathbf{K} \equiv \frac{d}{d\tau}\mathbf{p}_\mathrm{r} = \gamma \mathbf{F} = q(\gamma\,\mathbf{E} + \gamma\,\mathbf{v} \times \mathbf{B}) \ .$$

On the right-hand side we introduce the contravariant four-velocity $u^\mu$ (2.39):

$$u^\mu \equiv \gamma\,(c, \mathbf{v}) \equiv (u_0, \mathbf{u}) \ .$$

Then we have:

$$\mathbf{K} = q\left[u^0\left(\frac{1}{c}\mathbf{E}\right) + \mathbf{u} \times \mathbf{B}\right] \ .$$

We express the right-hand side by the contravariant field-strength tensor (2.115):

$$K_x = q\left[u^0\left(\frac{1}{c}E_x\right) + u_y B_z - u_z B_y\right] = q\left(u^0 F^{10} + u^2 F^{21} + u^3 F^{31}\right) = K^1 ,$$

$$K_y = q\left[u^0\left(\frac{1}{c}E_y\right) + u_z B_x - u_x B_z\right] = q\left(u^0 F^{20} + u^3 F^{32} + u^1 F^{12}\right) = K^2 ,$$

$$K_z = q\left[u^0\left(\frac{1}{c}E_z\right) + u_x B_y - u_y B_x\right] = q\left(u^0 F^{30} + u^1 F^{13} + u^2 F^{23}\right) = K^3 .$$

According to (2.114), if we replace in these expressions the terms $F^{\alpha\beta}$ with $\beta \neq 0$ by $-F^{\beta\alpha}$, and the contravariant by the covariant four-velocity ($u^0 \to u_0; \mathbf{u} \to -\mathbf{u}$), then because of $F^{\mu\mu} = 0$ it obviously holds:

$$K^1 = K_x = q\,F^{1\alpha}u_\alpha \, , \tag{2.159}$$

$$K^2 = K_y = q\,F^{2\alpha}u_\alpha \, , \tag{2.160}$$

$$K^3 = K_z = q\,F^{3\alpha}u_\alpha \, . \tag{2.161}$$

The right-hand sides transform certainly like the space components of a contravariant four-vector. It therefore suggests itself to complement the system by a corresponding time component:

$$\begin{aligned} K^0 &= q\,F^{0\alpha}u_\alpha \\ &= q\left[\left(-\frac{1}{c}E_x\right)(-u_x) + \left(-\frac{1}{c}E_y\right)(-u_y) + \left(-\frac{1}{c}E_z\right)(-u_z)\right] \\ &= q\left(\frac{1}{c}\mathbf{E}\right)\cdot(\gamma\,\mathbf{v}) \, . \end{aligned}$$

This expression is easy to interpret because

$$q\,\mathbf{v}\cdot\mathbf{E} = \mathbf{v}\cdot\mathbf{F} \, .$$

It holds namely with (2.53):

$$K^0 = \gamma\left(\frac{\mathbf{F}\cdot\mathbf{v}}{c}\right) = \gamma\frac{d}{dt}\left(\frac{1}{c}T_{\mathrm{r}}\right) = \frac{d}{d\tau}p^0 \, . \tag{2.162}$$

This result fits (2.157) and (2.158). We have therewith found the

**covariant representation of the Lorentz force :**

$$K^\mu \equiv \frac{d}{d\tau}p^\mu = q\,F^{\mu\alpha}u_\alpha = \gamma\left(\frac{1}{c}\mathbf{F}\cdot\mathbf{v}, \mathbf{F}\right) \, . \tag{2.163}$$

$K^\mu$ is the Minkowski force introduced in Sect. 2.2.2 for the covariant representation of classical mechanics. (2.163) documents once more the conclusiveness of the considerations in Sect. 2.2.2.

Let us investigate at the end once more, similar to that in Sect. 2.2.3, the explicit transformation behavior of the Lorentz force **F**. As usual, we consider for this purpose, two inertial systems $\Sigma$ and $\Sigma'$, out of which $\Sigma'$ moves relative to $\Sigma$ with the velocity **v** parallel to the $z$-axis. A particle of the charge $q$ with the velocity **u** in $\Sigma$ then experiences the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad \text{in } \Sigma \, . \tag{2.164}$$

Due to which force $\mathbf{F}'$ does it move as seen by an observer in $\Sigma'$? We determine $\mathbf{F}'$ from the transformation behavior of the space components of the contravariant Minkowski force (2.158):

$$K^{1'} = K^1 \, ; \quad K^{2'} = K^2 \, ,$$
$$K^{3'} = \gamma_v \left(-\beta \, K^0 + K^3\right) \, ; \quad \beta = \frac{v}{c} \, ; \quad \gamma_v = \left(1 - \beta^2\right)^{-1/2} \, .$$

If $\mathbf{u}'$ means the particle velocity in $\Sigma'$ then it follows from these equations:

$$\gamma_{u'} F'_x = \gamma_u F_x \, ; \quad \gamma_{u'} F'_y = \gamma_u F_y \, ;$$
$$\gamma_{u'} F'_z = \gamma_v \left(\gamma_u F_z - \beta \, \gamma_u \frac{\mathbf{F} \cdot \mathbf{u}}{c}\right) \, .$$

With the same considerations, which led in Sect. 2.2.3 to Eq. (2.80), one also finds here:

$$\gamma_{u'} = \gamma_u \gamma_v \left(1 - \frac{v \, u_z}{c^2}\right) \, .$$

Therewith, the components of the Lorentz force in $\Sigma'$ read:

$$F'_x = \frac{1}{\gamma_v} \frac{F_x}{1 - \frac{v \, u_z}{c^2}} \, , \tag{2.165}$$

$$F'_y = \frac{1}{\gamma_v} \frac{F_y}{1 - \frac{v \, u_z}{c^2}} \, , \tag{2.166}$$

$$F'_z = \frac{F_z - \frac{v}{c^2}(\mathbf{F} \cdot \mathbf{u})}{1 - \frac{v \, u_z}{c^2}} \, . \tag{2.167}$$

In Sect. 2.2.3 we had found identical formulae for the mechanical forces, which, however, because of (2.163), does not really surprise.

### 2.3.7 *Formulae of Relativistic Electrodynamics*

In the last section we have completed the covariant representation of the electrodynamics. For a better overview let us gather once more the most important formulae:

**electric charge:**

$$q: \text{ Lorentz invariant}$$

**four-current density:**

$$\begin{aligned} j^{\mu} = (c\rho, \mathbf{j}) = \; & \gamma\, \rho_0(c, \mathbf{v}) = \rho_0 u^{\mu} \\ & \rho_0 : \text{ rest-charge density ,} \\ & u^{\mu} \equiv \gamma(c, \mathbf{v}) : \text{ world-velocity .} \end{aligned}$$

**continuity equation :**

$$\partial_{\mu} j^{\mu} = 0 \; ; \quad \partial_{\mu} \equiv \left( \frac{1}{c} \frac{\partial}{\partial t}, \nabla \right) .$$

**four-potential:**

$$A^{\mu} \equiv \left( \frac{1}{c}\varphi, \mathbf{A} \right) .$$

**four-wave equation:**

$$\Box A^{\mu} = -\mu_0 j^{\mu} \; ; \quad \Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} .$$

**Lorenz gauge:**

$$\partial_{\mu} A^{\mu} = 0 .$$

**contravariant field-strength tensor:**

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \; ; \quad \partial^{\mu} \equiv \left( \frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right) ,$$

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & -B_z & B_y \\ \frac{1}{c}E_y & B_z & 0 & -B_x \\ \frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix} .$$

**covariant field-strength tensor:**

$$F_{\mu\nu} = \mu_{\mu\alpha}\mu_{\nu\beta}F^{\alpha\beta} \ .$$

**dual field-strength tensor:**

$$\overline{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

from $F_{\mu\nu}$ by substitution: $\mathbf{B} \longleftrightarrow -(1/c)\mathbf{E}$.

**Maxwell equations:**

**homogeneous:**

$$\partial_\alpha \overline{F}^{\alpha\beta} = 0 \ ; \quad \beta = 0, 1, 2, 3$$

or

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \ ,$$

$\alpha$, $\beta$, $\gamma$ arbitrary from $(0, 1, 2, 3)$.

**inhomogeneous:**

$$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta \ ; \quad \beta = 0, 1, 2, 3 \ .$$

**transformation of the fields:**

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c} :$$

$$\mathbf{E}' = \gamma\left[\mathbf{E} + c\,(\boldsymbol{\beta} \times \mathbf{B})\right] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E}) \ ,$$

$$\mathbf{B}' = \gamma\left[\mathbf{B} - \frac{1}{c}\,(\boldsymbol{\beta} \times \mathbf{E})\right] - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\,(\boldsymbol{\beta}\cdot\mathbf{B}) \ .$$

especially for $\mathbf{v} = v\,\mathbf{e}_z$:

$$B'_x = \gamma\left(B_x + \frac{\beta}{c}E_y\right) \quad E'_x = \gamma\left(E_x - \beta\,c\,B_y\right) \ ,$$

$$B'_y = \gamma\left(B_y - \frac{\beta}{c}E_x\right) \quad E'_y = \gamma\left(E_y + \beta\,c\,B_x\right) \ ,$$

$$B'_z = B_z \quad E'_z = E_z \ .$$

**invariants of the electromagnetic field:**

$$\left(\mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2\right) \quad \text{and} \quad \mathbf{E}\cdot\mathbf{B} \ .$$

**Minkowski force:**

$$K^{\mu} \equiv \gamma \left( \frac{\mathbf{F} \cdot \mathbf{v}}{c}, F_x, F_y, F_z \right) ,$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) : \text{ Lorentz force },$$

$$\mathbf{v} : \text{ particle velocity }.$$

**covariant force equation:**

$$K^{\mu} = m \frac{d}{d\tau} u^{\mu} = q F^{\mu\alpha} u_{\alpha} .$$

## 2.4 Covariant Lagrange Formulation

After having put into effect in Sect. 2.2 the relativistic generalization of the Newtonian mechanics, we now discuss, at least in the form of a set of statements, the relativistic Lagrangian mechanics (see Sect. 1, Vol. 2). In particular, the aim is to find a Lagrange function from which the equations of motion can be derived in the correct covariant form.

The most direct approach can be found by the use of the **Hamilton principle** (Sect. 1.3, Vol. 2):

*The* ***action functional*** *((1.120), Vol. 2)*

$$S = \int_{t_1}^{t_2} L\left(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\right) dt$$

*over the* ***Lagrange function*** $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ *takes among the competitive set* $M$ *of admitted paths ((1.118), Vol. 2)*

$$M = \{\mathbf{q}(t) : \ \mathbf{q}(t_1) = \mathbf{q}_i, \ \mathbf{q}(t_2) = \mathbf{q}_e\}$$

*an extreme value for the* ***real path****:*

$$\delta S = 0 .$$

The paths, which belong to $M$, have all the same initial and end configurations $\mathbf{q}_{\mathrm{i}}$ and $\mathbf{q}_{\mathrm{e}}$, which are adopted in each case at certain given times $t_1$ and $t_2$, while the various paths differ from one another by virtual displacements. One of the main consequences of this Hamilton principle are the **Lagrange equations of motion**:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 ; \quad j = 1, \ldots, s .$$

The *equivalence principle* (Sect. 1.3) can now be interpreted such that the action functional $S$, out of which the fundamental equations of motion arise, has to be **Lorentz invariant**. Furthermore, the Lagrange function $L$ should, in particular, be a four-scalar and should depend on the contravariant four-vectors $x^{\mu}$ and $u^{\mu}$ as well as the four-scalar $\tau$, instead of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $t$. We therefore try to formulate with the substitution

$$L \longrightarrow \overline{L}(x^{\mu}, u^{\mu}, \tau) \tag{2.168}$$

a

**covariant Hamilton principle**

$$\delta S = \delta \int_{\tau_1}^{\tau_2} d\tau\, \overline{L}(x^{\mu}, u^{\mu}, \tau) \overset{!}{=} 0\,, \tag{2.169}$$

out of which the covariant Lagrange equations arise:

$$\frac{d}{d\tau}\frac{\partial \overline{L}}{\partial u^{\mu}} - \frac{\partial \overline{L}}{\partial x^{\mu}} = 0\,, \tag{2.170}$$

However, the decisive question now is how to find the correct relativistic Lagrange function $\overline{L}$. As a rule one has to apply analogy-considerations to non-relativistic mechanics. In particular, the *familiar* relations should be reproduced for $v \ll c$. The main problem lies in the fact that because $L = T - V$, statements must be available about the potential $V$ and therewith about forces in covariant four-representation. Electromagnetic forces are, as we have seen in the last section, rather unproblematic, while, for instance, hardly any noteworthy theory exists about nuclear forces. Let us be content ourselves here with two important examples, which refer, as also from (2.168) to (2.170), to a single particle.

**(1) Force-free particle**

We look for a Lagrange function $\overline{L}_0$ from which we get the equation of motion:

$$m\frac{d}{d\tau}u^{\mu} = 0 \tag{2.171}$$

Non-relativistically one would investigate

$$S = \int_{t_1}^{t_2} L_0\left(\mathbf{x}(t), \mathbf{v}(t), t\right) dt\,,$$

where for the particle-momentum it then must hold

$$p_i = \frac{\partial L_0}{\partial v_i}\,; \qquad i = x, y, z\,.$$

If we understand in this relation by $p_i$ directly the relativistically correct expression,

$$\frac{\partial L_0}{\partial v_i} = p_{\mathrm{r}i} = \frac{m\,v_i}{\sqrt{1 - v^2/c^2}}\ ,$$

then it follows by integration:

$$L_0 = -m\,c^2 \sqrt{1 - v^2/c^2}\ . \tag{2.172}$$

This is the relativistically correct form. But one has to bear in mind that $L_0$ is no longer identical to the kinetic energy, thus differing from $T_{\mathrm{r}}$ in (2.54). For the corresponding Hamilton function, however, it holds ((2.8), Vol. 2):

$$H_0 = \sum_i p_{\mathrm{r}i} v_i - L_0 = \frac{m\,v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m\,c^2 \sqrt{1 - v^2/c^2} = \gamma\, m\, c^2$$

$$\Longrightarrow\quad H_0 = \frac{m\,c^2}{\sqrt{1 - v^2/c^2}} = T_{\mathrm{r}}\ . \tag{2.173}$$

We still substitute, eventually, in the action functional $S$ the time $t$ by the proper time $\tau$:

$$S = \int_{\tau_1}^{\tau_2} d\tau\, \gamma\, L_0\ .$$

The comparison with (2.169) yields:

$$\overline{L}_0 = \gamma\, L_0 = -m\,c^2\ . \tag{2.174}$$

We know from (2.41),

$$c^2 = u^\mu\, u_\mu\ ,$$

so that the correct functional dependence of the Lagrange function $\overline{L}_0$ on $u^\mu$ could be given by

$$\overline{L}_0 = \overline{L}_0\,(u^\mu) = -m u^\mu u_\mu\ , \tag{2.175}$$

or somewhat more generally:

$$\overline{L}_0 = -m c^\eta \left(u^\mu u_\mu\right)^{\frac{1}{2}(2-\eta)} \tag{2.176}$$

We determine $\eta$ by the requirement for *'correct'* equations of motion. Since $u^\mu$ is contravariant,

$$u'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} u^\alpha = L_{\mu\alpha} u^\alpha \ ,$$

the *velocity-gradient*

$$\frac{\partial}{\partial u'^\mu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial}{\partial u^\alpha} = \left(L^{-1}\right)_{\alpha\mu} \frac{\partial}{\partial u^\alpha} \tag{2.177}$$

transforms like a covariant four-vector. We define the **covariant canonical momentum**

$$p_\mu = -\frac{\partial \overline{L}}{\partial u^\mu} = \left(p^0, -\boldsymbol{p}\right) \tag{2.178}$$

with a minus sign in order to guarantee that the space components match the non-relativistic definition for $v \ll c$. We then have for the **contravariant canonical momentum** with (2.25), (2.24) and (2.19):

$$p^\mu = \mu^{\mu\alpha} p_\alpha = \left(p^0, \mathbf{p}\right) \tag{2.179}$$

Finally it should be valid for the **free** particle discussed here:

$$p_\mu^{(0)} = -\frac{\partial \overline{L}_0}{\partial u^\mu} \overset{!}{=} m u_\mu \iff p^\mu_{(0)} = m u^\mu \tag{2.180}$$

This means with respect to the ansatz (2.176):

$$\begin{aligned}
p_\mu^{(0)} &= -\frac{\partial}{\partial u^\mu} \left( -m c^\eta \left(u^\beta u_\beta\right)^{\frac{1}{2}(2-\eta)} \right) \\
&= m c^\eta \frac{\partial}{\partial u^\mu} \left(\mu_{\beta\gamma} u^\beta u^\gamma\right)^{\frac{1}{2}(2-\eta)} \\
&= (2-\eta) m c^\eta \mu_{\mu\gamma} u^\gamma \left(\mu_{\beta\gamma} u^\beta u^\gamma\right)^{-\frac{1}{2}\eta} \\
&= (2-\eta) m c^\eta u_\mu \left(u^\beta u_\beta\right)^{-\frac{1}{2}\eta} \\
&= (2-\eta) m u_\mu
\end{aligned}$$

The comparison with (2.180) leads to $\eta = 1$. That gives us the required functional dependency of the Lagrange function $\overline{L}_0$ on the four-velocity $u^\mu$:

$$\overline{L}_0 = -mc \left(u^\mu u_\mu\right)^{\frac{1}{2}} \tag{2.181}$$

The covariant Hamilton principle was the starting point for our considerations which led to the equations of motion (2.170). That means here for the force-free particle:

$$\frac{\partial \overline{L}_0}{\partial x^\mu} = 0 \quad \longrightarrow \quad \frac{d}{d\tau} p_\mu^{(0)} = m \frac{d}{d\tau} u_\mu = -\frac{d}{d\tau} \frac{\partial \overline{L}_0}{\partial u^\mu} = 0 \tag{2.182}$$

$$p_{(0)}^\mu = m \frac{d}{d\tau} u^\mu = 0 \tag{2.183}$$

The ansatz (2.181) for $\overline{L}_0$ appears to be correct.

**(2) Charged particle in the electromagnetic field**

We seek by this example, which was investigated elaborately in Sect. 1.2.3, Vol. 2 under the keyword *'generalized potentials'*, the covariant Lagrange function $\overline{L}$, from which the Lorentz-force equation,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \ ,$$

can be derived. Non-relativistically we have (see (1.79), Vol. 2):

$$L(\mathbf{x}, \mathbf{v}, t) = \frac{m}{2} v^2 + q(\mathbf{v} \cdot \mathbf{A}) - q\,\varphi \ . \tag{2.184}$$

$\varphi(\mathbf{x}, t)$ is the scalar potential and $\mathbf{A}(\mathbf{x}, t)$ the vector potential. The kinetic part should be identical to $L_0$ from example (1), while, supposedly, the two electromagnetic parts behave already relativistically correctly. With (2.172) for $L_0$ we use in the action functional

$$L = L_0 + q(\mathbf{v} \cdot \mathbf{A}) - q\,\varphi$$

and replace, as in example (1), the time $t$ by the proper time $\tau$. The comparison with (2.169) then leads to the following ansatz:

$$\begin{aligned} \overline{L} = \gamma\, L &= \gamma\, L_0 + q(\gamma\, \mathbf{v} \cdot \mathbf{A}) - q\,\gamma\,\varphi \\ &= \overline{L}_0 + q\left[\gamma\ \mathbf{v} \cdot \mathbf{A} - (\gamma c)\left(\frac{1}{c}\varphi\right)\right] \ . \end{aligned}$$

With (2.29), (2.40) and (2.107) we recognize on the right-hand side the scalar product of four-velocity $u^\mu$ and four-potential $A^\mu$:

$$\overline{L}\,(x^\mu, u^\mu, \tau) = \overline{L}_0\,(u^\mu) - q\left(u^\mu A_\mu\right) \ . \tag{2.185}$$

Let us test whether the known equation of motion results from this more or less *guessed* ansatz for $\overline{L}$. For this purpose we insert $\overline{L}$ into the covariant Lagrange equation (2.170):

$$\frac{d}{d\tau} \frac{\partial \overline{L}}{\partial u^\mu} \overset{(2.180)}{=} -m \frac{d}{d\tau} u_\mu - q \frac{d}{d\tau} A_\mu \overset{!}{=} \frac{\partial \overline{L}}{\partial x^\mu} = -q \frac{\partial}{\partial x^\mu} \left(u^\alpha A_\alpha\right) \ .$$

Hence it should hold:

$$m\frac{d}{d\tau}u_\mu = q\left[\partial_\mu\left(u^\alpha A_\alpha\right) - \frac{d}{d\tau}A_\mu\right] \stackrel{!}{=} K_\mu\ . \tag{2.186}$$

If our ansatz (2.185) indeed proves to be correct then we have found with (2.186), in a way as a by-product, a novel representation for the Minkowski force on a charged particle in the electromagnetic field.

We check at first the **space components**, for which we assume that they lead to the familiar Lorentz force. As usual we want to discuss the **contravariant** version of the four-vector (2.186)

$$K^\mu = q\left[\partial^\mu\left(u^\alpha A_\alpha\right) - \frac{d}{d\tau}A^\mu\right]\ . \tag{2.187}$$

With (2.31) and (2.38) we have then

$$K_i = q\left[-\frac{\partial}{\partial x_i}(-\gamma\ \mathbf{v}\cdot\mathbf{A} + \gamma\,\varphi) - \gamma\frac{d}{dt}A_i\right]\ , \qquad i \in (x, y, z)\ .$$

According to (2.158) it should thus follow for the Cartesian components of the Lorentz force:

$$F_i = \frac{1}{\gamma}K_i = q\left[\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A}) - \frac{\partial\varphi}{\partial x_i} - \frac{d}{dt}A_i\right]$$

Using

$$\frac{d}{dt}A_i = \mathbf{v}\cdot\nabla A_i + \frac{\partial A_i}{\partial t}\ ,$$

one can perform the following rearranging:

$$\begin{aligned}
(\mathbf{v}\times\mathrm{curl}\mathbf{A})_x &= v_y(\mathrm{curl}\mathbf{A})_z - v_z(\mathrm{curl}\mathbf{A})_y \\
&= v_y\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) - v_z\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) \\
&= \frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A}) - v_x\frac{\partial}{\partial x}A_x - v_y\frac{\partial}{\partial y}A_x - v_z\frac{\partial}{\partial z}A_x \\
&= \frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v}\cdot\nabla\ A_x \\
&= \frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A}) - \frac{d}{dt}A_x + \frac{\partial}{\partial t}A_x\ .
\end{aligned}$$

Analogously we calculate the other two components:

$$\frac{\partial}{\partial x_i}(\mathbf{v}\cdot\mathbf{A}) - \frac{d}{dt}A_i = (\mathbf{v}\times \mathrm{curl}\mathbf{A})_i - \frac{\partial}{\partial t}A_i \; .$$

This yields for the force components $F_i$:

$$F_i = q\left[-\frac{\partial\varphi}{\partial x_i} - \frac{\partial A_i}{\partial t} + (\mathbf{v}\times \mathrm{curl}\mathbf{A})_i\right] \; .$$

The first two summands just represent the $i$-th component of the electric field $\mathbf{E}$ ((4.21), Vol. 3), so that the space components of the four-force (2.187) lead indeed to the correct Lorentz force:

$$\mathbf{F} = q\left(-\nabla\varphi - \dot{\mathbf{A}} + \mathbf{v}\times \mathrm{curl}\mathbf{A}\right) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2.188}$$

It still remains to check the **time component:**

$$K^0 = q\left[\frac{1}{c}\,\frac{\partial}{\partial t}(-\gamma\;\mathbf{A}\cdot\mathbf{v} + \gamma\,\varphi) - \gamma\frac{d}{dt}\left(\frac{1}{c}\varphi\right)\right] \; .$$

We insert

$$\frac{d}{dt}\varphi = \nabla\varphi\cdot\mathbf{v} + \frac{\partial}{\partial t}\varphi$$

and obtain:

$$K^0 = \frac{1}{c}q\,\gamma\left(-\mathbf{v}\cdot\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi\cdot\mathbf{v}\right) = \frac{1}{c}q\;\gamma(\mathbf{E}\cdot\mathbf{v}) \; .$$

With (2.188) this can be written as

$$K^0 = \gamma\frac{\mathbf{F}\cdot\mathbf{v}}{c} \; . \tag{2.189}$$

We have therefore deduced from our ansatz (2.185) for $\overline{L}$, the according to (2.157) and (2.158) correct equation of motion:

$$K^\mu = m\frac{d}{d\tau}u^\mu = \gamma\left(\frac{\mathbf{F}\cdot\mathbf{v}}{c}, F_x, F_y, F_z\right) \; . \tag{2.190}$$

Our ansatz is therewith justified. Hence, we can conclude further on:

**canonical momentum**

$$p_\mu = -\frac{\partial\overline{L}}{\partial u^\mu} = m\,u_\mu + q\,A_\mu \; . \tag{2.191}$$

This is the covariant version; the contravariant vector arises simply by transposing the indexes:

$$p^{\mu} = m\,u^{\mu} + q\,A^{\mu} \ . \tag{2.192}$$

The **space components** ($i \in (x, y, z)$),

$$p_i = m\gamma\, v_i + q\,A_i \ , \tag{2.193}$$

contain, besides the relativistic mechanical momentum $p_{\mathrm{r}}^{(i)} = m\,\gamma\,v_i$, still an additional term $q\,A_i$, which, however, does **not** represent a relativistic effect, but appears already in the corresponding non-relativistic expression ((1.80), Vol. 2). One has therefore to distinguish here also canonical and mechanical momenta. We consider finally the **time component** of the canonical momentum:

$$p^0 = m\gamma c + q\frac{1}{c}\varphi = \frac{1}{c}\left(m\gamma c^2 + q\,\varphi\right) \ .$$

The first summand on the right-hand side is just the relativistic kinetic energy $T_{\mathrm{r}}$, which we introduced in (2.54). The bracket therefore represents the **total energy of the particle**:

$$E = T_{\mathrm{r}} + q\,\varphi = m\gamma c^2 + q\,\varphi \ . \tag{2.194}$$

For the **canonical momentum** it therewith holds :

$$p^{\mu} = \left(\frac{1}{c}E,\ m\gamma\,\mathbf{v} + q\mathbf{A}\right) \tag{2.195}$$

$$\equiv \left(\frac{1}{c}E,\ \mathbf{p}\right) \ . \tag{2.196}$$

From this one has to distinguish the **mechanical momentum**,

$$p_{\mathrm{m}}^{\mu} = \left(\frac{T_{\mathrm{r}}}{c}, m\,\gamma\,\mathbf{v}\right) = \left(\frac{T_{\mathrm{r}}}{c}, \mathbf{p}_{\mathrm{r}}\right) = \left[\frac{1}{c}(E - q\,\varphi),\ \mathbf{p} - q\,\mathbf{A}\right] \ , \tag{2.197}$$

as we have introduced it in (2.57). At the end we have to exploit (2.62),

$$p_{\mathrm{m}}^{\mu} p_{\mathrm{m}\mu} = m^2c^2 \ ,$$

i.e.,

$$m^2c^2 = -(\mathbf{p} - q\,\mathbf{A})^2 + \frac{1}{c^2}(E - q\,\varphi)^2 \ ,$$

in order to formulate as follows the

**relativistic energy of a charged particle in the electromagnetic field:**

$$E = \sqrt{(\mathbf{p} - q\,\mathbf{A})^2\,c^2 + m^2c^4} + q\,\varphi\ . \tag{2.198}$$

This expression is to be compared with (2.63).

## 2.5 Exercises

**Exercise 2.5.1** Show that the metric tensor of the Special Theory of Relativity (2.19) is a covariant tensor of second rank!

**Exercise 2.5.2**

1. Let $a^\mu$ be an arbitrary contravariant four-vector and $b_\mu$ any four-component vector. Thereby $a^\mu b_\mu$ is found to be always a Lorentz invariant. Show that then $b_\mu$ must be a covariant four-vector!
2. Let $T_{\mu\nu}$ be any tensor of second rank. $a^\mu$ and $c^\nu$ may be arbitrary contravariant four-vectors. The expression $T_{\mu\nu}a^\mu c^\nu$ may thereby always represent a Lorentz invariant. Prove that then $T_{\mu\nu}$ must be a covariant second-rank tensor.

**Exercise 2.5.3** Introduce by

$$b^\mu = \frac{d}{d\tau}u^\mu\ , \qquad u^\mu\ :\ \text{four-velocity}\ , \qquad \tau\ :\ \text{proper time}$$

the four-acceleration.

1. Show that the acceleration is in the Minkowski space always orthogonal to the velocity.
2. Express the components of $b^\mu$ explicitly by those of the system-velocity $\mathbf{v} = (v_x, v_y, v_z)$!

**Exercise 2.5.4** In order to guarantee that the astronaut in his rocket feels, during his flight through the universe, always quite *'as at home'*, it is arranged for that the acceleration of the rocket in the system $\Sigma'$, in which it is *momentarily at rest*, is constant and equal to the gravitational acceleration $g$. The initial velocity of the rocket at the time $t = 0$ shall be $v = 0$. Discuss in the following the flight in the *'earth-fixed'* system $\Sigma$, when the acceleration takes place in $z$-direction.

1. What is the velocity of the rocket $v(t)$ in the earth-system $\Sigma$ for $t > 0$?
2. Estimate by a non-relativistic calculation after what time the velocity of the rocket would exceed the velocity of light!
3. Calculate the time-dependent position of the rocket in the earth-system $\Sigma$!
4. How does the energy of the rocket change as a function of time?
5. Find an explicit connection between the proper time $\tau$ (*'age'*) of the astronaut and the time $t$ which has passed on earth!

**Exercise 2.5.5** Consider a rocket, on which any external forces do not act (gravitation-free space), and which is pushed by its mass expulsion. At the start the rocket has the mass $M_0$ and the velocity $v_0 = 0$, and to a later point in time the mass is $M$ and the velocity is $v$ relative to the earth-system (lab system). The mass $dm$ is expelled from the rocket with the velocity $v^*$ backwards. Calculate

1. non-relativistically,
2. relativistically

the velocity $v$ as function of the momentary mass $M$ of the rocket!

**Exercise 2.5.6** A particle moves in an inertial system $\Sigma$ with the velocity

$$\mathbf{u} = u(t)\,\mathbf{e}_z$$

along the $z$-axis. Show that its proper time,

$$\Delta\tau = \int_{\tau_1}^{\tau_2} d\tau = \int_{t_1}^{t_2} \frac{1}{\gamma_{u(t)}}\,dt \qquad\qquad \gamma_{u(t)} = \left(1 - \frac{u^2(t)}{c^2}\right)^{-\frac{1}{2}} ,$$

is a relativistic invariant.

**Exercise 2.5.7** The inertial system $\widehat{\Sigma}$ moves relative to the inertial system $\Sigma$ with the velocity $\mathbf{v}$ in arbitrary direction.

1. In $\widehat{\Sigma}$ the magnetic induction vanishes: $\widehat{\mathbf{B}} = 0$. Which connection exists then between $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$ in $\Sigma$? What can be said about $\mathbf{E}\cdot\mathbf{B}$ and $c^2\,\mathbf{B}^2 - \mathbf{E}^2$?
2. Answer the same questions for the case that in $\widehat{\Sigma}$ the electric field vanishes: $\widehat{\mathbf{E}} = 0$!

**Exercise 2.5.8** A particle with the charge $q$ moves in an inertial system $\Sigma$ with the velocity

$$\mathbf{u} = (a, a, a)$$

in a homogeneous magnetic field $\mathbf{B} = (B, 0, 0)$. Let $\Sigma'$ be an inertial system which moves relative to $\Sigma$ with the velocity $\mathbf{v} = v\,\mathbf{e}_z =$ const. Which forces are acting on the particle in $\Sigma$ and $\Sigma'$, respectively?

**Exercise 2.5.9** Show by use of the transformation formulae for the electromagnetic field that

$$\left(\mathbf{B} + \frac{\mathrm{i}}{c}\mathbf{E}\right)^2$$

is a Lorentz invariant.

**Exercise 2.5.10** Show by explicit calculation that the components of the dual field-strength tensor $\overline{F}^{\mu\nu}$ follow from those of the covariant tensor $F_{\mu\nu}$ by the substitution

$$\mathbf{B} \longleftrightarrow -\frac{1}{c}\mathbf{E} \,.$$

**Exercise 2.5.11** The inertial system $\widehat{\Sigma}$ moves relative to the inertial system $\Sigma$ with the velocity $\mathbf{v} = v\mathbf{e}_z =$ const. At the time $t = 0$ the two systems coincide. The point charge $q$ is located at the origin of $\widehat{\Sigma}$.

1. Determine the four-potentials $\hat{A}^{\mu}$ and $A^{\mu}$ in $\widehat{\Sigma}$ and $\Sigma$, respectively!
2. Calculate with $A^{\mu}$ the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of the point charge in $\Sigma$. Which connection exists between $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$? Compare the result for $(\mathbf{E}, \mathbf{B})$ with the formulae (2.142) and (2.143) which were derived from the transformation behavior of the field-strength tensor!
3. Does the scalar potential $\varphi$ fulfill the wave equation

$$\Box\,\varphi = -\frac{\rho}{\varepsilon_0}$$

   in $\Sigma$?

**Exercise 2.5.12** Let $\Sigma$, $\Sigma'$ be two inertial systems. The electromagnetic field in $\Sigma$ is $(\mathbf{E}, \mathbf{B})$ and in $\Sigma'$ $(\mathbf{E}', \mathbf{B}')$. The field $\mathbf{E}$ has the same direction in the whole space . $\Sigma'$ moves relative to $\Sigma$ with the velocity $\mathbf{v}_0$ parallel to $\mathbf{E}$, i.e., $\mathbf{v}_0 = \alpha\mathbf{E}$. A particle with the charge $q$ lies at the origin of coordinates in $\Sigma$. Use the Lorentz force on the particle in order to show that the component of $\mathbf{E}'$ in direction of $\mathbf{E}$ is equal to $E$.

**Exercise 2.5.13** Let $\Sigma$ and $\Sigma'$ be two inertial systems, where $\Sigma'$ moves relative to $\Sigma$ with the constant velocity $\mathbf{v} = v\mathbf{e}_z$. In $\Sigma$ a particle with charge $q$ and velocity $\mathbf{u} = (a, b, d)$ propagates in the electromagnetic field:

$$\mathbf{B} = (0, B, 0) \quad \mathbf{E} = \frac{1}{\sqrt{2}}(E, E, 0) \,.$$

Which forces act on the particle in $\Sigma$ and $\Sigma'$, respectively?

**Exercise 2.5.14**

1. A magnetic dipole is oriented parallel to the $z$-axis (*magnetic moment* $\mathbf{m} = m\,\mathbf{e}_z, m > 0$). What are the Cartesian components of the $\mathbf{B}$-field?
2. Calculate now the $\mathbf{E}$- and $\mathbf{B}$-field of a magnetic dipole which moves uniformly in straight-line in $z$-direction and whose moment is oriented parallel to the $z$-direction. The dipole at the time $t = 0$ is at the origin of $\Sigma$.
3. Rewrite the fields of the moving magnetic dipole by cylindrical coordinates!
4. Which shape do the electric field lines have in the $xy$-plane? How does the electric field change in course of time?
5. The $\mathbf{E}$-lines **do not** start or end at electric charges. Under which conditions can such electric fields occur?

**Exercise 2.5.15** Let $\Sigma$ and $\Sigma'$ be two inertial systems moving relative to one another with the velocity $\mathbf{v} = \text{const}$. Give qualitative arguments why in $\Sigma$ an electric field appears although in $\Sigma'$ only a $\mathbf{B}'$-field exists. Try to reason in the same manner why in $\Sigma$ a magnetic field occurs although in $\Sigma'$ only a pure electric field is present.

**Exercise 2.5.16** A particle of mass $m$ and the charge $q$ moves in a homogeneous magnetic field

$$\mathbf{B} = (0, 0, B) \ .$$

1. Show that its relativistic energy is constant in time.
2. Calculate the time-dependence of the relativistic momentum with the initial condition $\mathbf{v}_0 = (v_0, 0, 0)$.
3. Calculate the trajectory $\mathbf{r}(t)$ of the particle with $\mathbf{r}(t = 0) = (0, y_0, 0)$ where $y_0 = \gamma \frac{m\,v_0}{q\,B}$.

**Exercise 2.5.17** Let a charged particle (charge $q$, mass $m$) be moving in a homogeneous electric field

$$\mathbf{E} = (E, 0, 0)$$

with the initial conditions

$$\mathbf{r}(t = 0) = (0, 0, z_0) \ ; \quad \mathbf{v}(t = 0) = (0, v_0, 0) \ .$$

1. Calculate the time-dependence of the relativistic kinetic energy $T_\mathrm{r} = T_\mathrm{r}(t)$.
2. Determine the velocity of the particle $\mathbf{v} = \mathbf{v}(t)$.
3. How does the path $\mathbf{r}(t)$ of the particle look like?

**Exercise 2.5.18** The *'rest mass'* $m(0)$ of an electron, expressed in MeV, amounts to 0,511 MeV. Let the electron be accelerated by a voltage of 200 kV.

1. Calculate the *'mass-increase'* (2.59) $m(v)/m(0)$. (Note the remarks in Sect. 2.2.2 on the velocity-dependence of the mass).
2. Express the velocity $v$ in units of $c$.
3. Calculate the percentage error that appears in the calculation of $v$, if one uses for the kinetic energy the non-relativistic expression $T = \frac{1}{2} m(0) v^2$.

**Exercise 2.5.19** Consider the elastic collision of two particles with the same mass $m$ in the inertial system $\Sigma$ in which one of the two particles is at rest *before* the collision. The other particle shall have, before the collision, the energy $T_\mathrm{r}$ and the momentum $\mathbf{p}_\mathrm{r}$. After the collision the two particles possess the energy $T_{\mathrm{r}1}$ and $T_{\mathrm{r}2}$, respectively, and the momentum $\mathbf{p}_{\mathrm{r}1}$ and $\mathbf{p}_{\mathrm{r}2}$, respectively.

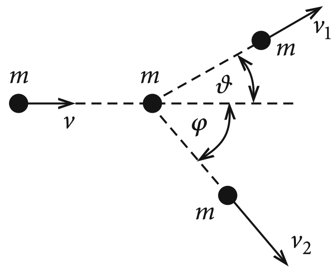



**Fig. 2.5** Relativistic collision of two particles with equal masses $m$, out of which one is at rest before the collision

1. Calculate the angle $\vartheta$ between the momenta $\mathbf{p}_{r1}$ and $\mathbf{p}_{r2}$ after the collision as functions of $T_r$ and $T_{r1}$!
2. Discuss the limiting cases $v \ll c$ and $v \approx c$, where $v$ is the velocity of the (moving) particle before the collision in the system $\Sigma$.

**Exercise 2.5.20** A $\pi^+$-meson *at rest* decays within $2.5 \cdot 10^{-8}$ s into a $\mu^+$-meson and a neutrino. Let the $\pi^+$-meson have a kinetic energy which is equal to its rest energy.

1. What is the velocity of the $\pi^+$-meson?
2. Which distance is covered by the meson before its decay, detected by an observer*'at rest'*?

**Exercise 2.5.21** In the case of a non-relativistic collision of two particles with equal masses the trajectories enclose after the collision an angle of $\pi/2$. Show that for the relativistic collision it holds (see notations in Fig. 2.5)

$$\tan\varphi \, \tan\vartheta = \frac{2}{\gamma + 1}$$

$\left(\gamma = \left(1 - v^2/c^2\right)^{-1/2}\right)$. Demonstrate that $\vartheta + \varphi \leq \pi/2$, where the equality sign becomes valid in the non-relativistic limit ($\gamma \to 1$).

**Exercise 2.5.22** Let a photon (rest mass $m = 0$) have the same momentum as a 1 MeV-electron. What is the energy of the photon?

## 2.6 Self-Examination Questions

### To Section 2.1

1. What is to be understood by the covariant formulation of a physical law?
2. Are the Newtonian laws form-invariant under Lorentz transformations?
3. How many components does a tensor of zeroth (first, second) rank have?

4. How does a contravariant (covariant) four-vector behave under a Lorentz transformation?
5. Define the three different types of second-rank tensors!
6. Which type of second-rank tensor represents in the strict sense a matrix?
7. What does the *contraction of a tensor* mean? How does the rank of a tensor change with such a *contraction*?
8. How is the scalar product of four-vectors defined?
9. How can one *convert* a covariant into a contravariant tensor?
10. How are gradient, divergence, and d'Alembert operator written in the four-dimensional version?

## To Section 2.2

1. How is the differential $(d\tau)^2$ of the proper time connected with the differential length square $(ds)^2$?
2. How is the world-(four-)velocity $u^\mu$ defined?
3. Which physical meaning does the *norm* of the world-velocity have?
4. Which considerations lead to which *ansatz* for the relativistic generalization of the Newton's law of inertia?
5. Which analogy-considerations lead to the space components of the four-momentum and the Minkowski force?
6. How does the Minkowski force $K^\mu$ look like?
7. Calculate the scalar product $K^\mu u^\mu$ of Minkowski force and world-velocity!
8. Which physical meaning does the time component of the Minkowski force have?
9. Comment on the relativistic expression of the kinetic energy $T_\mathrm{r}$. How does it look like for $v \ll c$?
10. What does one understand by the rest energy of a mass point?
11. Which meaning does the time component of the four-momentum have?
12. Which simple relation exists between the rest energy and the norm $p^\mu p_\mu$ of the four-momentum?
13. Which relationship exists in the Special Theory of Relativity between the conservation laws of momentum and energy?
14. Explain the equivalence of mass and energy!
15. Which physical process allows for a direct derivation of the relativistic momentum $\mathbf{p}_\mathrm{r}$ and the relativistic kinetic energy $T_\mathrm{r}$?
16. How do $\mathbf{p}_\mathrm{r}$ and $T_\mathrm{r}$ behave under a Lorentz transformation?

## To Section 2.3

1. What is the reason why charge density $\rho$ and current density $\mathbf{j}$ are not Lorentz invariant, in contrast to the charge $q$ itself?
2. How are charge density and current density influenced by a change of the inertial system $\Sigma_0 \xrightarrow{v} \Sigma$?
3. How is the four-current density $j^\mu$ defined?

4. Which meaning does the divergence $\partial_\mu j^\mu$ of the four-current density have?
5. How is the continuity equation written in covariant four-dimensional form?
6. What are the components of the electromagnetic four-potential $A^\mu$?
7. Formulate the covariant wave equations of the electromagnetic potentials! How are they related to the Maxwell equations?
8. How does the Lorenz gauge read in covariant form?
9. Which relation exists in the Minkowski space between the electric field **E** and the magnetic induction **B**?
10. Why can the field-strength tensor be seen as four-dimensional generalization of the curl of $A^\mu$?
11. How do the covariant and the contravariant field-strength tensor differ?
12. Give a Lorentz invariant of the electromagnetic field!
13. Is it possible by use of a Lorentz transformation to transfer a pure **B**-field into a pure **E**-field?
14. How can the Maxwell equations be expressed by the field-strength tensor? Demonstrate the covariance!
15. What does one understand by the dual field-strength tensor?
16. By which substitution of the fields does one get from the covariant to the dual field-strength tensor?
17. How can one represent the homogeneous Maxwell equations by the dual field-strength tensor?
18. How is the scalar product of **E** and **B** influenced by a change of the inertial system?
19. Is the electric field of a moving point charge radial in the rest system of the observer? Is it isotropic?
20. What is valid for the amplitude $E(\beta)/E(0)$ of the electric field of a point charge, in the direction of the motion ($\varphi = 0, \pi$) and perpendicular to that ($\varphi = \pi/2$), respectively?
21. How is the covariant formulation of the Lorentz force found by the use of the field-strength tensor?

## To Section 2.4

1. What does the equivalence principle mean for the action functional $S$?
2. Formulate the covariant Hamilton principle!
3. How are the covariant Lagrange equations of motion to be read?
4. What is the relativistically correct form of the Lagrange function $\overline{L}_0$ for a force-free particle?
5. How can the canonical momentum be derived from the relativistic Lagrange function?
6. Give reasons for the covariant Lagrange function $\overline{L}$ of a particle in the electromagnetic field!
7. Express the Minkowski force $K^\mu$ of a particle in the electromagnetic field by the world-velocity $u^\mu$ and the four-potential $A^\mu$.

8. Give the time component of the Minkowski force which acts on a particle in the electromagnetic field!
9. Discuss the difference between the canonical and the mechanical momentum of a particle in the electromagnetic field.
10. How is the time component of the canonical momentum of a particle in the electromagnetic field related to its total energy $E$?
11. Formulate the relativistic total energy of a charged particle in the electromagnetic field!

# Appendix A
# Solutions of the Exercises

## Section 1.6

**Solution 1.6.1** Let $\Sigma$ and $\Sigma'$ be two inertial systems which move in $z$-direction relative to one another with the velocity $v = 0.8\,c$. Let $\Sigma$ be the rest system of the earth, $\Sigma'$ that of the space craft:

$$\Sigma \xrightarrow{v} \Sigma' \,.$$

The origins of coordinates coincide exactly when the spaceship has the distance $d$ from the earth. (Let the spaceship be at the origin of $\Sigma'$.) According to (1.20) and (1.21):

$$z' = \gamma(z - v\,t) \; ; \quad t' = \gamma\left(t - \frac{v}{c^2}z\right) \; .$$

In $\Sigma$ the signal is emitted at the space-time point

$$z_0 = -d \; ; \quad t_0 = 0 \; ,$$

and in $\Sigma'$ at:

$$z'_0 = -\gamma\,d \; ; \quad t'_0 = \gamma\frac{v}{c^2}d \; .$$

The signal has in $\Sigma'$ the velocity $c$ and reaches the ship after the lapse of time:

$$\Delta t' = \frac{\gamma\,d}{c} \quad \text{(solution for 2.).}$$



W. Nolting, *Theoretical Physics 4*, DOI 10.1007/978-3-319-44371-3

The *arrival of the signal* has in $\Sigma$ the coordinates:

$$
\begin{aligned}
z_1 &= \text{ position of the ship at the time } t_1 \ , \\
z_1 &= v\, t_1 \ .
\end{aligned}
$$

We look for $t_1$. In $\Sigma'$ it holds for the point $(z_1, t_1)$:

$$
\begin{aligned}
z_1' &= \gamma(z_1 - v\, t_1) = 0 \ , \\
t_1' &= \gamma\left(t_1 - \frac{v}{c^2} z_1\right) = \gamma\, t_1 \left(1 - \frac{v^2}{c^2}\right) = \frac{t_1}{\gamma} \ .
\end{aligned}
$$

The transit time, seen from the ship, thus amounts to:

$$
\Delta t' = t_1' - t_0' = \frac{t_1}{\gamma} - \frac{\gamma d}{c^2} v \ .
$$

We equalize the two expressions for $\Delta t'$ and solve for $t_1$:

$$
t_1 = \frac{\gamma^2 d}{c}\left(1 + \frac{v}{c}\right) = \frac{d}{c - v} \ .
$$

Since $t_0 = 0$ it follows for the transit time, measured on the earth-station:

$$
\Delta t = t_1 - t_0 = \frac{d}{c - v} \quad \text{(solution for 1.)} \ .
$$

Numerical values:

$$
\begin{aligned}
\gamma &= \left(1 - (0.8)^2\right)^{-1/2} = \frac{5}{3} = 1.667 \\
&\implies \Delta t = 3700\,\mathrm{s} \ , \\
\Delta t' &= 11.100\,\mathrm{s} \ .
\end{aligned}
$$

Observed from $\Sigma'$ the signal reaches the spaceship at an earth-distance of

$$
\begin{aligned}
\Delta z' &= d + v\, \Delta t' \\
&= \left(6.66 \cdot 10^8 + 26.64 \cdot 10^8\right)\ \mathrm{km} \\
&= 3.33 \cdot 10^9\,\mathrm{km} \ .
\end{aligned}
$$

**Solution 1.6.2**

$$
\begin{aligned}
&v = \frac{3}{5}c \implies \gamma = \frac{5}{4}\,,\\
&x = x' = 10\,\text{m}\,; \quad y = y' = 15\,\text{m}\,,\\
&z = \gamma\left(z' + v\,t'\right) = \frac{5}{4}\left(20 + \frac{9}{5}\,4\right)\text{m} = 34\,\text{m}\,,\\
&t = \gamma\left(t' + \frac{v}{c^2}z'\right) = \frac{5}{4}\left(4 + \frac{1}{5}20\right)10^{-8}\,\text{s} = 1\cdot 10^{-7}\,\text{s}\,.
\end{aligned}
$$

**Solution 1.6.3**

$$
\begin{aligned}
t_1' = t_2' &\iff \gamma\left(t_1 - \frac{v}{c^2}z_1\right) = \gamma\left(t_2 - \frac{v}{c^2}z_2\right)\\
&\iff \frac{v}{c^2}(z_2 - z_1) = t_2 - t_1\\
&\iff v = c^2\,\frac{t_2 - t_1}{z_2 - z_1} = c^2\,\frac{-\frac{1}{2}\frac{z_0}{c}}{z_0}\\
&\iff v = -\frac{1}{2}c\,.
\end{aligned}
$$

The time in $\Sigma'$ results from:

$$
\begin{aligned}
t' = \gamma\left(t_1 - \frac{v}{c^2}z_1\right) &= \frac{1}{\sqrt{1-\frac{1}{4}}}\left(\frac{z_0}{c} + \frac{1}{2}\frac{z_0}{c}\right)\\
&\implies t' = \sqrt{3}\,\frac{z_0}{c}\,.
\end{aligned}
$$

**Solution 1.6.4** It holds in $\Sigma$:

$$
z_1 = z_2\,; \quad t_2 - t_1 = 4\,\text{s}\,.
$$

$\Sigma'$ moves with the velocity $v$ relative to $\Sigma$. We determine $v$ by:

$$
\begin{aligned}
&t_2' - t_1' = \gamma\left[t_2 - t_1 - \frac{v}{c^2}\left(z_2 - z_1\right)\right] = \gamma\,4\,\text{s} \stackrel{!}{=} 5\,\text{s}\\
&\implies \gamma = \frac{5}{4} \implies \gamma^{-2} = \frac{16}{25} = \left(1 - \frac{v^2}{c^2}\right)\\
&\implies v = \frac{3}{5}c\,.
\end{aligned}
$$

This yields the spatial distance in $\Sigma'$:

$$\begin{aligned} z_2' - z_1' &= \gamma\left[z_2 - z_1 - v\left(t_2 - t_1\right)\right] \\ &= \gamma\, v\left(t_1 - t_2\right) = (-4\,\mathrm{s})\frac{5}{4}\,\frac{3}{4}\,3\cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}} \\ &= -\,0.9\cdot 10^9\,\mathrm{m}\;. \end{aligned}$$

**Solution 1.6.5** It is measured in $\Sigma$:

$$t_1 = t_2\;;\quad z_2 - z_1 = 3\,\mathrm{km}\;.$$

In $\Sigma'$:

$$z_2' - z_1' = 5\,\mathrm{km}\;.$$

This means:

$$\begin{aligned} 5\,\mathrm{km} &= \left[z_2 - z_1 - v\left(t_2 - t_1\right)\right]\,\gamma = \gamma\,3\,\mathrm{km} \\ \Longrightarrow\;\; \gamma &= \frac{5}{3} \;\;\Longrightarrow\;\; v = \frac{4}{5}c\;. \end{aligned}$$

For the time distance we get therewith:

$$\begin{aligned} t_2' - t_1' &= -\gamma\frac{v}{c^2}\left(z_2 - z_1\right) \\ &= -\frac{5}{3}\,\frac{4}{5}\,\frac{1}{3}\cdot 10^{-5}\,\frac{\mathrm{s}}{\mathrm{km}}\,3\,\mathrm{km} = -\frac{4}{3}\cdot 10^{-5}\,\mathrm{s}\;. \end{aligned}$$

**Solution 1.6.6** Relative velocity $\mathbf{v}$ between inertial systems $\Sigma$ and $\Sigma'$ in arbitrary space direction.

$$\mathbf{r}:\quad \text{position vector in } \Sigma\;.$$

Decomposition:

$$\begin{aligned} \mathbf{r}_\parallel &= \frac{1}{v^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v}: \quad \text{component parallel to } \mathbf{v}\;, \\ \mathbf{r}_\perp &= \mathbf{r} - \mathbf{r}_\parallel: \quad \text{component perpendicular to } \mathbf{v}\;. \end{aligned}$$

Analogous decomposition in $\Sigma'$:

$$\mathbf{r}' = \mathbf{r}_\parallel' + \mathbf{r}_\perp\;;\quad \mathbf{r}_\parallel' = \frac{1}{v^2}\left(\mathbf{r}'\cdot\mathbf{v}\right)\mathbf{v}\;.$$

The perpendicular component remains unchanged under the transformation:

$$\mathbf{r}'_\perp = \mathbf{r}_\perp = \mathbf{r} - \frac{1}{v^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v}\ .$$

We exploit the isotropy of the space which permits us to rotate the $z$-axis into the direction of $\mathbf{v}$. The subsequent argumentation corresponds to that of the special case $\mathbf{v} = v\,\mathbf{e}_z$, with which (1.16) has been derived:

$$r'_\parallel = \frac{1}{v}\left(\mathbf{r}'\cdot\mathbf{v}\right) = \gamma\left(\frac{1}{v}(\mathbf{r}\cdot\mathbf{v}) - vt\right)$$

$$t' = \gamma\left(t - \frac{\beta}{c}\underbrace{\frac{1}{v}(\mathbf{r}\cdot\mathbf{v})}_{r_\parallel}\right)\ .$$

It remains altogether:

$$\mathbf{r}' = \mathbf{r} - \gamma\,\mathbf{v}\,t + \frac{\gamma-1}{v^2}(\mathbf{r}\cdot\mathbf{v})\mathbf{v}$$

$$t' = \gamma\left(t - \frac{\beta}{c}\frac{1}{v}(\mathbf{r}\cdot\mathbf{v})\right)\ .$$

Transformation matrix:

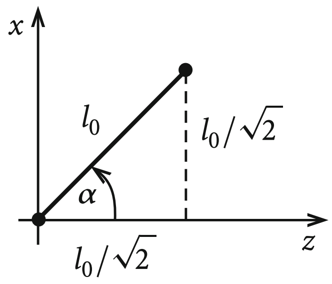
$$u_{x,\,y,\,z} = \frac{v_{x,\,y,\,z}}{v}\ , \qquad \beta = \frac{v}{c}$$

$$\hat{L} \equiv \begin{pmatrix} \gamma & -\beta\gamma u_x & -\beta\gamma u_y & -\beta\gamma u_z \\ -\gamma\frac{v_x}{c} & 1+(\gamma-1)u_x^2 & (\gamma-1)u_x u_y & (\gamma-1)u_x u_z \\ -\gamma\frac{v_y}{c} & (\gamma-1)u_y u_x & 1+(\gamma-1)u_y^2 & (\gamma-1)u_y u_z \\ -\gamma\frac{v_z}{c} & (\gamma-1)u_z u_x & (\gamma-1)u_z u_y & 1+(\gamma-1)u_z^2 \end{pmatrix}\ .$$

Special case: $\quad \mathbf{v} = v\,\mathbf{e}_x \quad \Longrightarrow \quad u_x = 1\ , \quad u_y = u_z = 0$

$$\hat{L} \equiv \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ .$$

**Fig. A.1**




## Solution 1.6.7

1. In $\Sigma'$ the length contraction (1.28) takes place in $z$-direction (Fig. A.1):

$$\Sigma:\ \mathbf{l}_0 = \left(\tfrac{1}{\sqrt{2}}l_0\,,\ \tfrac{1}{\sqrt{2}}l_0\right)\ ,$$
$$\Sigma':\mathbf{l}_0' = \left(\tfrac{1}{\sqrt{2}}l_0\,,\ \tfrac{1}{\gamma}\tfrac{1}{\sqrt{2}}l_0\right)\ .$$

This means:

$$\tan\alpha' = \frac{l'_{0x}}{l'_{0z}} = \frac{\frac{1}{\sqrt{2}}l_0}{\frac{1}{\gamma}\frac{1}{\sqrt{2}}l_0} = \gamma$$
$$\Longrightarrow\ \alpha' = \arctan\gamma\ ;\quad \gamma = \left(1-\frac{v^2}{c^2}\right)^{-(1/2)}\ .$$

2. $\mathbf{u} \neq \mathbf{u}(t)$

$$\Longrightarrow\ \tan\alpha = \frac{u_x}{u_z} = \frac{v}{2v} = \frac{1}{2}\ \Longrightarrow\ \alpha = \arctan\frac{1}{2}\ .$$

It holds in $\Sigma'$ ((1.36), (1.38)):

$$u_x' = \frac{1}{\gamma}\frac{u_x}{1-\frac{v\,u_z}{c^2}}\ ;\quad u_z' = \frac{u_z - v}{1-\frac{v\,u_z}{c^2}}$$
$$\Longrightarrow\ \tan\alpha' = \frac{u_x'}{u_z'} = \frac{1}{\gamma}\frac{u_x}{u_z - v} = \frac{1}{\gamma}\frac{v}{2v-v} = \frac{1}{\gamma}$$
$$\Longrightarrow\ \alpha' = \arctan\frac{1}{\gamma}\ .$$

3. The photon moves with the velocity of light:

$$\Longrightarrow \quad u_x = \frac{1}{\sqrt{2}}c\,; \quad u_z = \frac{1}{\sqrt{2}}c$$

$$\Longrightarrow \quad \tan\alpha' = \frac{1}{\gamma}\frac{u_x}{u_z - v} = \frac{c}{\gamma\left(c - \sqrt{2}v\right)}$$

$$\Longrightarrow \quad \alpha' = \arctan\left[\frac{c\sqrt{1-\frac{v^2}{c^2}}}{c - \sqrt{2}\,v}\right] .$$

**Solution 1.6.8**

$$\Sigma : \quad \text{rest system of the observer}$$

$$\Sigma' : \quad \text{rest system of the rocket .}$$

Choose the system of coordinates such that the two origins coincide with $P_0$ at $t = t' = 0$.

1. $t'_{\mathrm{a}} = 0\,; \quad t'_{\mathrm{e}} = \frac{L_0}{c}$.
2. $\Sigma$ moves relative to $\Sigma'$ with $(-v)$:

$$z = \gamma\left(z' + v\,t'\right)\;; \quad t = \gamma\left(t' + \frac{v}{c^2}z'\right)\;,$$

$$t_{\mathrm{a}} = 0\;,$$

$$t_{\mathrm{e}} = \gamma\left(t'_{\mathrm{e}} + \frac{v}{c^2}z'_{\mathrm{e}}\right) = \gamma\left[\frac{L_0}{c} + \frac{v}{c^2}\left(-L_0\right)\right]$$

$$= \gamma\frac{L_0}{c}(1-\beta) = \frac{L_0}{c}\sqrt{\frac{(1-\beta)^2}{1-\beta^2}}$$

$$\Longrightarrow \quad t_{\mathrm{e}} = \frac{L_0}{c}\sqrt{\frac{1-\beta}{1+\beta}}\;.$$

3. $z_0 = 0 \;\Longrightarrow\; z'_0 + v\,t'_0 = 0$

   $z'_0$: position of the end of the rocket in $\Sigma'$, i.e.:

$$z'_0 = -L_0 \;\Longrightarrow\; t'_0 = \frac{L_0}{v}\;.$$

   $t_0$ is looked for:

$$t_0 = \gamma\left(t'_0 + \frac{v}{c^2}z'_0\right) = \gamma\frac{L_0}{v}(1-\beta^2) = \frac{L_0}{\gamma\,v}\;.$$

**Solution 1.6.9** The formulae (1.36) to (1.38) follow for the velocity components according to the Lorentz transformation. These are to be evaluated for the given special case:

$$u_x = 0\,, \quad u_y = c\,, \quad u_z = 0 \quad \Longrightarrow \quad u'_x = 0\,, \quad u'_y = \frac{1}{\gamma}c\,, \quad u'_z = -v$$

$$\Longrightarrow \quad \mathbf{u}' = \sqrt{\frac{c^2}{\gamma^2} + v^2} = \sqrt{c^2} = c$$

$$\mathbf{u}' = \left(0, \frac{1}{\gamma}c, -v\right) = c\left(0, \frac{1}{\gamma}, -\beta\right)\,.$$

**Solution 1.6.10**

(a) $\mathbf{x}_1 - \mathbf{x}_2 = (-3\,\mathrm{m}, 0, -4\,\mathrm{m})$

$$\Longrightarrow \quad |\mathbf{x}_1 - \mathbf{x}_2|^2 = 25\,\mathrm{m}^2\,,$$

$$c^2\,(t_1 - t_2)^2 = 9\cdot 10^{16}\,\frac{\mathrm{m}^2}{\mathrm{s}^2} 9\cdot 10^{-16}\,\mathrm{s}^2 = 81\,\mathrm{m}^2$$

$\Longrightarrow$ The space-time interval

$$s_{12}^2 = c^2\,(t_1 - t_2)^2 - |\mathbf{x}_1 - \mathbf{x}_2|^2 = 56\,\mathrm{m}^2 > 0$$

is *time-like*. A causal correlation is therefore possible! Simultaneity, however, is not reachable by any Lorentz transformation.

(b)

$$\mathbf{x}_1 - \mathbf{x}_2 = (3\,\mathrm{m}, -5\,\mathrm{m}, -5\,\mathrm{m})$$

$$\Longrightarrow \quad |\mathbf{x}_1 - \mathbf{x}_2|^2 = 59\,\mathrm{m}^2\,,$$

$$c^2\,(t_1 - t_2)^2 = 9\cdot 10^{16}\frac{\mathrm{m}^2}{\mathrm{s}^2}\,4\cdot 10^{-16}\,\mathrm{s}^2 = 36\,\mathrm{m}^2\,.$$

The space-time interval

$$s_{12}^2 = 36\,\mathrm{m}^2 - 59\,\mathrm{m}^2 = -23\,\mathrm{m}^2 < 0$$

is *space-like*. It is therefore **no** causal correlation possible. In contrast, simultaneity is reachable.

$$\beta^2 = \frac{c^2\,(t_1 - t_2)^2}{|\mathbf{x}_1 - \mathbf{x}_2|^2} = \frac{36}{59} = 0.61\,.$$

The inertial system $\Sigma'$ has to move with the velocity

$$v = 0.781\,c = 2.343 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$$

in the direction of $(\mathbf{x}_1 - \mathbf{x}_2)$, in order to let the events appear simultaneously in $\Sigma'$.

**Solution 1.6.11**

1. Time distance, seen from the earth:

$$\Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}\ .$$

The lifetime of the muon $\tau$ in its own rest system amounts in the earth-system to:

$$\Delta t_\tau = \frac{\tau}{\sqrt{1-\frac{v^2}{c^2}}}\ .$$

In order that the muon reaches the surface of the earth, we must have:

$$\Delta t_\tau \overset{!}{\geq} \frac{H}{v}\ .$$

$v$ here is the velocity of the muon.

$$\begin{aligned}\varepsilon = \frac{c-v}{c} \quad &\curvearrowright \quad \frac{v}{c} = 1-\varepsilon \\ \curvearrowright \quad \frac{v^2}{c^2} &= 1 - 2\varepsilon + \varepsilon^2 \approx 1 - 2\varepsilon \quad \curvearrowright \quad 1 - \frac{v^2}{c^2} \approx 2\varepsilon\ .\end{aligned}$$

It remains the requirement:

$$\begin{aligned}\Delta t_\tau &\approx \frac{\tau}{\sqrt{2\varepsilon}} \overset{!}{\geq} \frac{H}{v} \\ &\curvearrowright \varepsilon \leq \frac{1}{2}\left(\frac{\tau v}{H}\right)^2 \leq \frac{1}{2}\left(\frac{\tau c}{H}\right)^2 \\ &= \frac{1}{2}\left(\frac{2\cdot 10^{-6}\,\mathrm{s}\cdot 3\cdot 10^{10}\,\frac{\mathrm{cm}}{\mathrm{s}}}{30\cdot 10^5\,\mathrm{cm}}\right)^2 = \frac{1}{2}\left(2\cdot\frac{10^4}{10^6}\right)^2 = 2\cdot 10^{-4}\ .\end{aligned}$$

It follows therewith:

$$v = c(1-\varepsilon) \geq c\left(1 - 2\cdot 10^{-4}\right) = 0.998\,c\ .$$

2. Because of the length contraction, the distance to the earth, as seen from the muon, amounts to:

$$H' = H\sqrt{1-\frac{v^2}{c^2}} \approx H\sqrt{2\varepsilon}$$
$$\leq 30\cdot 10^3\,\mathrm{m}\cdot\sqrt{4\cdot 10^{-4}} = 600\,\mathrm{m}\,.$$

The distance is thus less than 600 m, which the muon is approximately able to cover within its lifetime!

**Solution 1.6.12** Before the interference the rays traverse a rectangle but in opposite directions. That allows for a very accurate measurement of the velocity of light in a medium which itself moves with the velocity $v$.

- $v_1 = \pm v$:
  velocity of the flowing liquid relative to the *'at rest'* laboratory frame.
- $v_2 = \pm\frac{c}{n}$:
  velocity of light in the medium, i.e. relative to the flowing liquid.
- $v_3$:
  sought-after velocity of the light relative to the laboratory frame.

**parallel paths of rays:**

**upper:** $\Sigma_1 \xrightarrow{v} \Sigma_2 \xrightarrow{\frac{c}{n}} \Sigma_3\,.$

Addition theorem of velocities:

$$v_3^{\mathrm{p}} = \frac{v+\frac{c}{n}}{1+\frac{v}{nc}} \approx \left(v+\frac{c}{n}\right)\left(1-\frac{v}{nc}\right) \approx v+\frac{c}{n}-\frac{v}{n^2}\,.$$

Here $v \ll c$ was exploited. It therefore holds in the upper part of the tube:

$$\left|v_3^{\mathrm{p}}\right| \approx \frac{c}{n} + v\left(1-\frac{1}{n^2}\right)\,.$$

For the lower part of the tube we have:

**lower:** $\Sigma_1 \xrightarrow{-v} \Sigma_2 \xrightarrow{-\frac{c}{n}} \Sigma_3\,.$

Addition theorem of velocities:

$$v_3^{\mathrm{p}} = \frac{-v-\frac{c}{n}}{1+\frac{v}{nc}} \approx -\left(v+\frac{c}{n}\right)\left(1-\frac{v}{nc}\right) \approx -v-\frac{c}{n}+\frac{v}{n^2}\,.$$

That yields the same magnitude of velocity:

$$\left|v_3^{\mathrm{p}}\right| \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) .$$

The time needed in both parts of the tube in the case of parallel paths of rays thus amounts to:

$$t_{\mathrm{p}} = \frac{2l}{\frac{c}{n} + v\left(1 - \frac{1}{n^2}\right)} .$$

**antiparallel paths of rays:**

$$\textbf{upper:} \qquad \Sigma_1 \xrightarrow{v} \Sigma_2 \xrightarrow{-\frac{c}{n}} \Sigma_3 .$$

Addition theorem of velocities:

$$v_3^{\mathrm{ap}} = \frac{v - \frac{c}{n}}{1 - \frac{v}{nc}} \approx \left(v - \frac{c}{n}\right)\left(1 + \frac{v}{nc}\right) \approx v - \frac{c}{n} - \frac{v}{n^2} = -\frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) .$$

Here again $v \ll c$ was used. It therefore holds in the upper part of the tube in the case of antiparallel paths of rays:

$$\left|v_3^{\mathrm{ap}}\right| \approx \frac{c}{n} - v\left(1 - \frac{1}{n^2}\right) .$$

For the lower part of the tube we have:

$$\textbf{lower:} \qquad \Sigma_1 \xrightarrow{-v} \Sigma_2 \xrightarrow{\frac{c}{n}} \Sigma_3 .$$

That leads with the addition theorem of velocities to:

$$v_3^{\mathrm{ap}} = \frac{-v + \frac{c}{n}}{1 - \frac{v}{nc}} \approx \left(-v + \frac{c}{n}\right)\left(1 + \frac{v}{nc}\right) \approx -v + \frac{c}{n} + \frac{v}{n^2} = \frac{c}{n} - v\left(1 - \frac{1}{n^2}\right) .$$

That yields the same magnitude of velocity as in the upper part:

$$\left|v_3^{\mathrm{ap}}\right| \approx \frac{c}{n} - v\left(1 - \frac{1}{n^2}\right) .$$

The time needed in both parts of the tube in the case of antiparallel paths of rays thus amounts to:

$$t_{\mathrm{ap}} = \frac{2l}{\frac{c}{n} - v\left(1 - \frac{1}{n^2}\right)} .$$

That results in a difference of the transit times:

$$\Delta t = t_{\mathrm{ap}} - t_{\mathrm{p}} = 2l \left( \frac{1}{\frac{c}{n} - f v} - \frac{1}{\frac{c}{n} + f v} \right) .$$

The Fresnel's dragging coefficient is therewith calculated to be:

$$f = 1 - \frac{1}{n^2} .$$

In the case $n = 1.33$ for water it turns out to be a surely measurable effect. The agreement of theory and experiment proves the correctness of the Lorentz transformation, here demonstrated for the addition theorem of velocities!

## Section 2.5

**Solution 2.5.1** Definition equation for the metric tensor is given in (2.18):

$$(ds)^2 = \mu_{\alpha\beta}\, dx^\alpha\, dx^\beta .$$

'Lowering' of an index:

$$\mu_{\alpha\beta}\, dx^\beta = dx_\alpha .$$

Therewith one realizes that

$$(ds)^2 = dx^\alpha\, dx_\alpha = (dx, dx)$$

is a scalar product and thus a Lorentz invariant. That is then also true for

$$\mu_{\alpha\beta}\, dx^\alpha\, dx^\beta .$$

Consider the change of the inertial system:

$$\Sigma \longrightarrow \widehat{\Sigma}$$

It must be required:

$$\hat{\mu}_{\alpha\beta}\, d\hat{x}^\alpha\, d\hat{x}^\beta \stackrel{!}{=} \mu_{\alpha\beta}\, dx^\alpha\, dx^\beta .$$

Otherwise it holds:

$$\hat{\mu}_{\alpha\beta}\, d\hat{x}^{\alpha}\, d\hat{x}^{\beta} = \hat{\mu}_{\alpha\beta}\, \frac{\partial \hat{x}^{\alpha}}{\partial x^{\mu}}\, dx^{\mu}\, \frac{\partial \hat{x}^{\beta}}{\partial x^{\lambda}}\, dx^{\lambda} = \hat{\mu}_{\mu\lambda}\, \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, \frac{\partial \hat{x}^{\lambda}}{\partial x^{\beta}}\, dx^{\alpha}\, dx^{\beta} \ .$$

In the second step we could interchange the indexes because of the summation convention. The comparison yields:

$$\mu_{\alpha\beta} = \hat{\mu}_{\mu\lambda}\, \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, \frac{\partial \hat{x}^{\lambda}}{\partial x^{\beta}} \ .$$

Transformation behavior:

$$\frac{\partial x^{\alpha}}{\partial \hat{x}^{\rho}}\, \frac{\partial x^{\beta}}{\partial \hat{x}^{\sigma}}\, \mu_{\alpha\beta} = \hat{\mu}_{\mu\lambda}\, \frac{\partial x^{\alpha}}{\partial \hat{x}^{\rho}}\, \frac{\partial x^{\beta}}{\partial \hat{x}^{\sigma}}\, \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, \frac{\partial \hat{x}^{\lambda}}{\partial x^{\beta}} = \hat{\mu}_{\mu\lambda}\, \delta_{\rho\mu}\, \delta_{\sigma\lambda} = \hat{\mu}_{\rho\sigma} \ .$$

That is the transformation behavior of a covariant second-rank tensor!

**Solution 2.5.2** We use the familiar notation and regard the summation convention:

$$\Sigma \xrightarrow{\hat{L}} \widehat{\Sigma} \ .$$

1. Let $a^{\mu}$ be a contravariant four-vector $\curvearrowright$

$$\hat{a}^{\mu} = \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, a^{\alpha} = L_{\mu\alpha} a^{\alpha} \ .$$

$x^{\mu}$ is the position vector in the Minkowski space. Let now $b_{\mu}$ be a four-component vector with at first unknown transformation behavior. Then:

$$\begin{aligned}
\hat{a}^{\mu}\hat{b}_{\mu} &= \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, a^{\alpha}\hat{b}_{\mu} \\
&= a^{\alpha}\, \frac{\partial \hat{x}^{\mu}}{\partial x^{\alpha}}\, \hat{b}_{\mu} \\
&= a^{\mu}\, \frac{\partial \hat{x}^{\alpha}}{\partial x^{\mu}}\, \hat{b}_{\alpha} \\
&\overset{!}{=} a^{\mu} b_{\mu} \qquad \text{(Lorentz invariant).}
\end{aligned}$$

In the third step we have simply interchanged the indexes $\alpha$ and $\mu$, which is of course allowed because of the summation convention. Since $a^{\mu}$ is arbitrary it follows already

$$b_{\mu} = \frac{\partial \hat{x}^{\alpha}}{\partial x^{\mu}}\, \hat{b}_{\alpha} \ .$$

Transformation behavior:

$$\frac{\partial x^\mu}{\partial \hat{x}^\beta} b_\mu = \frac{\partial x^\mu}{\partial \hat{x}^\beta} \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \hat{b}_\alpha = \underbrace{(\hat{L}^{-1})_{\mu\beta} L_{\alpha\mu}}_{\delta_{\alpha\beta}} \hat{b}_\alpha \ .$$

Hence it holds:

$$\hat{b}_\beta = \frac{\partial x^\mu}{\partial \hat{x}^\beta} b_\mu \ .$$

$b_\mu$ therefore transforms like a covariant four-vector.

2. We calculate:

$$\begin{aligned}
\hat{T}_{\mu\nu} \hat{a}^\mu \hat{c}^\nu &= \hat{T}_{\mu\nu} \frac{\partial \hat{x}^\mu}{\partial x^\alpha} a^\alpha \frac{\partial \hat{x}^\nu}{\partial x^\beta} c^\beta \\
&= \frac{\partial \hat{x}^\mu}{\partial x^\alpha} \frac{\partial \hat{x}^\nu}{\partial x^\beta} \hat{T}_{\mu\nu} a^\alpha c^\beta \\
&= \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \hat{T}_{\alpha\beta} a^\mu c^\nu \quad \text{(indexes } \mu, \alpha \text{ and } \nu, \beta \text{ interchanged)} \\
&\overset{!}{=} T_{\mu\nu} a^\mu c^\nu \quad \text{(Lorentz invariant!)}
\end{aligned}$$

$a^\mu$ and $c^\nu$ arbitrary $\curvearrowright$

$$T_{\mu\nu} = \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \hat{T}_{\alpha\beta} \ .$$

Transformation behavior:

$$\frac{\partial x^\mu}{\partial \hat{x}^\gamma} \frac{\partial x^\nu}{\partial \hat{x}^\delta} T_{\mu\nu} = \frac{\partial x^\mu}{\partial \hat{x}^\gamma} \frac{\partial x^\nu}{\partial \hat{x}^\delta} \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \hat{T}_{\alpha\beta} = \delta_{\alpha\gamma} \delta_{\delta\beta} \hat{T}_{\alpha\beta} = \hat{T}_{\gamma\delta} \ .$$

$T_{\mu\nu}$ is therefore a covariant tensor of second rank.

**Solution 2.5.3** According to (2.40) we have for the four-velocity:

$$u^\mu = \gamma(v)(c, \mathbf{v}) \ .$$

The proper time $\tau$ is Lorentz invariant. Hence

$$b^\mu = \frac{d}{d\tau} u^\mu$$

is a contravariant four-vector.

(a)

$$u_\mu u^\mu = \gamma^2 \left(c^2 - \mathbf{v}^2\right) = c^2 \; .$$

It follows:

$$\frac{d}{d\tau} u_\mu u^\mu = 0 = \mu_{\mu\nu} \frac{d}{d\tau} \left(u^\nu u^\mu\right) \; .$$

$\mu_{\mu\nu}$ is the metric tensor. Note the summation convention. It therefore holds:

$$0 = 2\, \mu_{\mu\nu}\, u^\nu \frac{d}{d\tau} u^\mu = 2\, u_\mu \frac{d}{d\tau} u^\mu = 2\, u_\mu b^\mu \; .$$

This is the assertion:

$$(u, b) = u_\mu b^\mu = 0 \; .$$

(b)

$$b^\mu = \frac{d}{d\tau} u^\mu = \frac{du^\mu}{dt} \frac{dt}{d\tau} \; .$$

According to (2.38):

$$dt = \frac{d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\, d\tau \; ,$$

and therewith

$$b^\mu = \gamma \frac{du^\mu}{dt} \; .$$

We use

$$\begin{aligned} \frac{dv}{dt} &= \frac{d}{dt} \sqrt{v_x^2 + v_y^2 + v_z^2} \\ &= \frac{1}{v} \left(v_x \dot{v}_x + v_y \dot{v}_y + v_z \dot{v}_z\right) = \frac{\mathbf{v}}{v} \cdot \frac{d\mathbf{v}}{dt} \end{aligned}$$

for the calculation of

$$\frac{d}{dt} \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \frac{v}{c^2} \frac{dv}{dt} = (\gamma(v))^3 \frac{1}{c^2} \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}\right) \; .$$

By insertion we get the components of the acceleration:

$$b^0 = \gamma \frac{d}{dt}(\gamma c) = \frac{1}{c}\left(1 - \frac{v^2}{c^2}\right)^{-2}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) ,$$

$$b^1 = \gamma \frac{d}{dt}(\gamma v_x) = \frac{\frac{dv_x}{dt}}{1-\frac{v^2}{c^2}} + \frac{v_x}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-2}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) ,$$

$$b^2 = \gamma \frac{d}{dt}(\gamma v_y) = \frac{\frac{dv_y}{dt}}{1-\frac{v^2}{c^2}} + \frac{v_y}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-2}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) ,$$

$$b^3 = \gamma \frac{d}{dt}(\gamma v_z) = \frac{\frac{dv_z}{dt}}{1-\frac{v^2}{c^2}} + \frac{v_z}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-2}\left(\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\right) .$$

**Solution 2.5.4**

1. Momentary rest system of the rocket $\Sigma'$ (see solution to Exercise 2.5.3 for $v'_x = v'_y = v'_z = 0$; $dv'_z/dt = g$):

$$b'^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g \end{pmatrix}$$

$\Sigma' \to \Sigma$ by Lorentz transformation of the contravariant four vector

$$b^{\mu} = L_{\mu\lambda}\, b'^{\lambda} \quad \Longrightarrow \quad b_z = b^3 = \gamma g = \frac{g}{\sqrt{1 - v^2/c^2}}$$

$$v = v(t): \qquad \text{momentary velocity of the rocket } (\sim \mathbf{e}_z) .$$

It is also valid

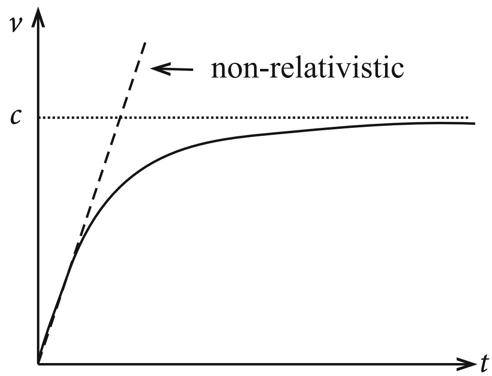
$$b^3 = \frac{d}{d\tau}u_z = \gamma \frac{d}{dt}\gamma\, v_z$$

$$(v_z = v) \quad \Longrightarrow \quad \gamma g = \gamma \frac{d}{dt}\gamma\, v \quad \Longrightarrow \quad g = \frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right) .$$

After integration:

$$gt = \frac{v(t)}{\sqrt{1 - \frac{v^2(t)}{c^2}}} .$$

**Fig. A.2**




Solving for $v(t)$:

$$v(t) = c\frac{gt/c}{\sqrt{1+(gt/c)^2}} \quad\Longrightarrow\quad v \underset{t\to\infty}{\longrightarrow} c\,.$$

2. Approximately after 1 year. Non-relativistically one finds (Fig. A.2):

$$v = gt \simeq 9.8\,\frac{\text{m}}{\text{s}^2}\cdot 86.400\,\text{s}\cdot 365 \approx 3\cdot 10^8\,\frac{\text{m}}{\text{s}}\,.$$

3.

$$\begin{aligned} v(t) &= \frac{dz}{dt} \qquad \text{with} \qquad z(t=0)=0\,,\\ z(t) &= \frac{c^2}{g}\left(\sqrt{1+(gt/c)^2}-1\right)\,.\\ gt \ll c: &\qquad z(t) \approx \frac{1}{2}gt^2\,,\\ gt \gg c: &\qquad z(t) \approx ct\,. \end{aligned}$$

4.

$$T_\text{r} = \frac{mc^2}{\sqrt{1-v^2/c^2}}\,,$$

$$\sqrt{1-v^2/c^2} = \frac{1}{\sqrt{1+(gt/c)^2}} \quad\Longrightarrow\quad T_\text{r} = mc^2\sqrt{1+(gt/c)^2}\,.$$

5. Proper-time interval $d\tau$:

$$d\tau = \sqrt{1 - v^2/c^2}\, dt \; ; \qquad \sqrt{1 - v^2/c^2} = \frac{1}{\sqrt{1 + (gt/c)^2}} \; .$$

Integration:

$$\tau = \frac{c}{g} \ln \left( \frac{gt}{c} + \sqrt{1 + (gt/c)^2} \right) \; .$$

Reversal:

$$t = \frac{c}{g} \sinh \left( \frac{g\,\tau}{c} \right) \; .$$

Numerical example:

| $\tau$ (years) | $t$ (years) |
|---|---|
| 4 | 30 |
| 8 | 1855.6 |
| 40 | $3.98 \cdot 10^{17}$ |
| 100 | $2.95 \cdot 10^{44}$ |

**Solution 2.5.5**

1. **non-relativistic**:
   (Momentary) law of momentum in the **frame of laboratory**:

$$Mv = (M - dm)(v + dv) + dm(v - v^*) \; .$$

To the left the momentum before, to the right the momentum after the expulsion. We neglect quadratic terms of the differentials:

$$Mv \approx Mv + M\,dv - v^* dm \;\curvearrowright\; M\,dv = v^* dm \; .$$

Let $m = M_0 - M$ be the mass of the so far not yet ejected fuel, i.e., $dm = -dM$. Then it remains to be integrated:

$$\frac{dM}{M} = -\frac{1}{v^*}\, dv$$

$$\int_{M_0}^{M} d\ln M' = -\frac{1}{v^*} \int_0^v dv'$$

with the result:

$$v = v^* \ln \frac{M_0}{M} \ . \tag{A.1}$$

For $M \to 0$ the velocity of the rocket grows indefinitely.

alternatively:

Law of conservation of momentum in the **center-of-mass system**:

$$0 = (M - dm)\, dv_\mathrm{r} - v^* dm \ .$$

$dv_\mathrm{r}$ is thereby the velocity of the rocket relative to its (own) rest system, produced by the expulsion. If one neglects again quadratic terms of the differentials then it remains:

$$M\, dv_\mathrm{r} \approx v^* dm \ .$$

Now it holds, however, according to Galilei: $v + dv = v + dv_\mathrm{r}$. Thus it is $dv_\mathrm{r} = dv$. We obtain therefore the same differential equation as above, with the same result (A.1).

2. **relativistic**

   It is recommendable to use the **center-of-mass system**, i.e., the momentary rest system of the rocket.

   Relativistic law of momentum:

$$0 = \gamma_r (M - dm)\, dv_\mathrm{r} - \gamma^* d\bar{m}\, v^*$$
$$\gamma_r = \left(1 - \left(\frac{dv_\mathrm{r}}{c}\right)^2\right)^{-\frac{1}{2}} \qquad \gamma^* = \left(1 - \left(\frac{v^*}{c}\right)^2\right)^{-\frac{1}{2}} \ .$$

Note that $dm \neq d\bar{m}$, since a part of the ejected mass $dm$ is needed for the kinetic energy for the flying away mass $d\bar{m}$. Quadratic differentials shall again be neglected, i.e. in particular: $\gamma_r \approx 1$. Then we have:

$$M\, dv_\mathrm{r} = \frac{d\bar{m}\, v^*}{\sqrt{1 - \left(\frac{v^*}{c}\right)^2}} \ . \tag{A.2}$$

We still need $d\bar{m}$. For this purpose we use the conservation of energy law:

$$Mc^2 = \frac{(M - dm)\,c^2}{\sqrt{1 - \left(\frac{dv_r}{c}\right)^2}} + \frac{d\bar{m}\,c^2}{\sqrt{1 - \left(\frac{v^*}{c}\right)^2}}$$

$$\approx (M - dm)\,c^2 + \frac{d\bar{m}\,c^2}{\sqrt{1 - \left(\frac{v^*}{c}\right)^2}}$$

$$\curvearrowright \; dm = \frac{d\bar{m}}{\sqrt{1 - \left(\frac{v^*}{c}\right)^2}} \, . \tag{A.3}$$

As expected $dm > d\bar{m}$, since a part of the mass is converted into kinetic energy. Insertion of (A.3) into (A.2) yields with

$$M\,dv_r = v^*\,dm \tag{A.4}$$

a conditional equation which formally holds non-relativistically also. However, for the transition from the center-of-mass system to the laboratory system the addition theorem of velocities has to be obeyed. In the laboratory system the velocity change $v + dv$ is observed:

$$v + dv \cong \frac{v + dv_r}{1 + \frac{v\,dv_r}{c^2}} \approx (v + dv_r)\left(1 - \frac{v\,dv_r}{c^2}\right)$$

$$\approx v + dv_r - \frac{v^2}{c^2}dv_r = v + (1 - \beta^2)\,dv_r$$

$$\curvearrowright \; dv \approx (1 - \beta^2)\,dv_r \, .$$

Hence it is $dv < dv_r$! Insertion into (A.4) yields:

$$\frac{M\,dv}{1 - \beta^2} = v^*\,dm \, .$$

Let $m = M_0 - M$ again be the mass of the so far ejected fuel, then it follows with $dm = -dM$ the conditional equation

$$\frac{dv}{1 - \beta^2} = -v^*\,\frac{dM}{M} \, .$$

This can be easily integrated:

$$\int_0^v \frac{dv'}{1 - \beta'^2} = \frac{c}{2}\int_0^\beta (d\ln(1 + \beta) - d\ln(1 - \beta)) = -v^* \int_{M_0}^M d\ln M$$

with the result:

$$\frac{c}{2}\,\ln\frac{1+\beta}{1-\beta} = v^*\,\ln\frac{M_0}{M}. \tag{A.5}$$

In the non-relativistic limit $\beta \ll 1$ this yields because of

$$\ln\frac{1+\beta}{1-\beta} = \ln(1+\beta) - \ln(1-\beta) \approx 2\beta$$

just the result (A.1). In contrast, it follows with

$$\frac{1+\beta}{1-\beta} = \left(\frac{M_0}{M}\right)^{\frac{2v^*}{c}}$$

relativistically correct:

$$v = c\,\frac{1-\left(\frac{M}{M_0}\right)^{\frac{2v^*}{c}}}{1+\left(\frac{M}{M_0}\right)^{\frac{2v^*}{c}}}. \tag{A.6}$$

Now $v$ does not grow indefinitely for $M \to 0$ but saturates at the limiting velocity $c$.

### Solution 2.5.6

$$\Sigma \xrightarrow{v\,\boldsymbol{e}_z} \widehat{\Sigma} \qquad \text{(inertial system)}$$

To be shown:

$$\int_{t_1}^{t_2} \frac{1}{\gamma_{u(t)}}\,dt \stackrel{!}{=} \int_{\hat{t}_1}^{\hat{t}_2} \frac{1}{\gamma_{\hat{u}(\hat{t})}}\,d\hat{t}\,.$$

$\hat{u} = \hat{u}(\hat{t})$ is here the velocity of the particle in $\widehat{\Sigma}$ and

$$\gamma_{\hat{u}(\hat{t})} = \left(1 - \frac{\hat{u}^2(\hat{t})}{c^2}\right)^{-\frac{1}{2}}\,.$$

Transformation formulae:

$$\hat{z} = \gamma_v(z(t) - v\,t) \;\curvearrowright\; d\hat{z} = \gamma_v(u-v)\,dt$$

$$\hat{t} = \gamma_v\left(t - \frac{v}{c^2}\,z(t)\right) \;\curvearrowright\; d\hat{t} = \gamma_v\left(1 - \frac{v}{c^2}\,u\right)\,dt$$

$$\curvearrowright\; \hat{u} = \frac{d\hat{z}}{d\hat{t}} = \frac{u-v}{1-\frac{v}{c^2}u}\,.$$

The Special Theory of Relativity is valid for all points of the world line of the particle, and therefore these formulae apply accordingly! We now substitute $\hat{t} = \hat{t}(t)$ with

$$dt = \frac{1}{\gamma_v} \frac{1}{1 - \frac{v}{c^2}\, u(t)}\, d\hat{t}$$

and calculate:

$$\begin{aligned}
\int_{\hat{t}_1}^{\hat{t}_2} \frac{1}{\gamma_{\hat{u}}}\, d\hat{t} &= \int_{\hat{t}_1}^{\hat{t}_2} \sqrt{1 - \frac{\hat{u}^2(\hat{t})}{c^2}}\, d\hat{t} \\
&= \int_{t_1}^{t_2} \left(1 - \frac{1}{c^2} \frac{(u(t) - v)^2}{\left(1 - \frac{u(t)}{c^2} v\right)^2}\right)^{\frac{1}{2}} \gamma_v \left(1 - \frac{v}{c^2}\, u(t)\right) dt \\
&= \int_{t_1}^{t_2} \left(\left(1 - \frac{v}{c^2}\, u(t)\right)^2 - \frac{1}{c^2}\,(u(t) - v)^2\right)^{\frac{1}{2}} \gamma_v\, dt \\
&= \int_{t_1}^{t_2} \left(1 + \frac{v^2}{c^4}\, u^2(t) - \frac{2v}{c^2}\, u(t) - \frac{1}{c^2}\, u^2(t) - \frac{v^2}{c^2} + \frac{2u(t)v}{c^2}\right)^{\frac{1}{2}} \gamma_v\, dt \\
&= \int_{t_1}^{t_2} \left(\left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{1}{c^2}\, u^2(t)\right)\right)^{\frac{1}{2}} \gamma_v\, dt \\
&= \int_{t_1}^{t_2} \left(1 - \frac{1}{c^2}\, u^2(t)\right)^{\frac{1}{2}} dt \\
&= \int_{t_1}^{t_2} \frac{1}{\gamma_u}\, dt\,.
\end{aligned}$$

From that it follows indeed that the proper time is an invariant, as already shown in a different manner in Sect. 2.2.1!

**Solution 2.5.7** For $\Sigma \rightarrow \widehat{\Sigma}$ the transformation formulae (2.142) and (2.143) are applicable:

$$\begin{aligned}
\widehat{\mathbf{E}} &= \gamma\left(\mathbf{E} + c(\boldsymbol{\beta} \times \mathbf{B})\right) - \frac{\gamma^2}{1+\gamma} \boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \mathbf{E}) \\
\widehat{\mathbf{B}} &= \gamma\left(\mathbf{B} - \frac{1}{c}(\boldsymbol{\beta} \times \mathbf{E})\right) - \frac{\gamma^2}{1+\gamma} \boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \mathbf{B})\,; \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}\,.
\end{aligned}$$

For $\widehat{\Sigma} \to \Sigma$, i.e. $\boldsymbol{\beta} \to -\boldsymbol{\beta}$, it is then:

$$\mathbf{E} = \gamma\left(\widehat{\mathbf{E}} - c(\boldsymbol{\beta} \times \widehat{\mathbf{B}})\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \widehat{\mathbf{E}})$$
$$\mathbf{B} = \gamma\left(\widehat{\mathbf{B}} + \frac{1}{c}(\boldsymbol{\beta} \times \widehat{\mathbf{E}})\right) - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \widehat{\mathbf{B}})\ .$$

1. $\widehat{\mathbf{B}} \equiv 0$

$$\mathbf{E} = \gamma\widehat{\mathbf{E}} - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \widehat{\mathbf{E}})$$
$$\mathbf{B} = \frac{\gamma}{c}(\boldsymbol{\beta} \times \widehat{\mathbf{E}})\ .$$

It follows therewith:

$$\boldsymbol{\beta} \times \mathbf{E} = \gamma(\boldsymbol{\beta} \times \widehat{\mathbf{E}}) = c\mathbf{B}$$
$$\curvearrowright\ \mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})\ .$$

Lorentz invariants (Sect. 2.3.5):

$$\mathbf{E} \cdot \mathbf{B} \ \text{ and } \ c^2\mathbf{B}^2 - \mathbf{E}^2\ .$$

Therewith it follows from $\widehat{\mathbf{E}} \cdot \widehat{\mathbf{B}} = 0$:

$$\mathbf{E} \cdot \mathbf{B} = 0$$

and from

$$c^2\widehat{\mathbf{B}}^2 - \widehat{\mathbf{E}}^2 = -\widehat{\mathbf{E}}^2 < 0$$

one finds

$$c^2\mathbf{B}^2 - \mathbf{E}^2 < 0\ .$$

2. $\widehat{\mathbf{E}} \equiv 0$

$$\curvearrowright\ \mathbf{E} = -\gamma c(\boldsymbol{\beta} \times \widehat{\mathbf{B}})$$
$$\mathbf{B} = \gamma\widehat{\mathbf{B}} - \frac{\gamma^2}{1+\gamma}\boldsymbol{\beta}\,(\boldsymbol{\beta} \cdot \widehat{\mathbf{B}})\ .$$

This means,

$$\boldsymbol{\beta} \times \mathbf{B} = \gamma(\boldsymbol{\beta} \times \widehat{\mathbf{B}}) = \gamma \left(-\frac{1}{\gamma c}\right) \mathbf{E} \, ,$$

and therewith

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} \, .$$

Lorentz invariants:

$$\widehat{\mathbf{E}} \cdot \widehat{\mathbf{B}} = 0 \;\curvearrowright\; \mathbf{E} \cdot \mathbf{B} = 0$$
$$c^2 \widehat{\mathbf{B}}^2 - \widehat{\mathbf{E}}^2 = c^2 \widehat{\mathbf{B}}^2 > 0 \;\curvearrowright\; c^2 \mathbf{B}^2 - \mathbf{E}^2 > 0 \, .$$

**Solution 2.5.8** It holds in $\Sigma$:

$$\mathbf{F} = q \, \mathbf{u} \times \mathbf{B} = q(0, a\,B, -a\,B) = q\,a\,B(0, 1, -1) \, .$$

In $\Sigma'$ one finds (2.165)–(2.167):

$$F'_x = \frac{1}{\gamma} \frac{F_x}{1 - \frac{v\,u_z}{c^2}} = 0 \, , \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \, ,$$
$$F'_y = \frac{1}{\gamma} \frac{F_y}{1 - \frac{v\,u_z}{c^2}} = \frac{1}{\gamma} \frac{q\,a\,B}{1 - \frac{v\,a}{c^2}} \, .$$

$\mathbf{F} \cdot \mathbf{u} = q\,a^2\,B(1 - 1) = 0$:

$$F'_z = \frac{F_z - \frac{v}{c^2}(\mathbf{F} \cdot \mathbf{u})}{1 - \frac{v\,u_z}{c^2}} = \frac{-q\,a\,B}{1 - \frac{v\,a}{c^2}}$$
$$\Longrightarrow \mathbf{F}' = \frac{q\,a\,B}{1 - \frac{v\,a}{c^2}} \left(0, \frac{1}{\gamma}, -1\right) \, .$$

**Solution 2.5.9** Let $\Sigma$ and $\Sigma'$ be two arbitrary inertial systems with

$$\Sigma \xrightarrow{v} \Sigma' \quad (\mathbf{v} = v\,\mathbf{e}_z) \, .$$

We show that

$$\left(\mathbf{B}' + \frac{i}{c}\mathbf{E}'\right)^2 \stackrel{!}{=} \left(\mathbf{B} + \frac{i}{c}\mathbf{E}\right)^2$$

The transformation formulae (2.132) to (2.137) yield:

$$B'_x + \frac{i}{c}E'_x = \gamma\left(B_x + \frac{i}{c}E_x\right) - i\,\gamma\,\beta\left(B_y + \frac{i}{c}E_y\right)\,,$$
$$B'_y + \frac{i}{c}E'_y = \gamma\left(B_y + \frac{i}{c}E_y\right) + i\,\gamma\beta\left(B_x + \frac{i}{c}E_x\right)\,,$$
$$B'_z + \frac{i}{c}E'_z = B_z + \frac{i}{c}E_z\,.$$

From that we get:

$$\begin{aligned}\left(\mathbf{B}' + \frac{i}{c}\mathbf{E}'\right)^2 &= \left(B_x + \frac{i}{c}E_x\right)^2\gamma^2(1-\beta^2)\\ &\quad + \left(B_y + \frac{i}{c}E_y\right)^2\gamma^2(1-\beta^2) + \left(B_z + \frac{i}{c}E_z\right)^2\\ &= \left(\mathbf{B} + \frac{i}{c}\mathbf{E}\right)^2\,.\end{aligned}$$

**Solution 2.5.10** Because of $\overline{F}^{\mu\nu} = -\overline{F}^{\nu\mu}$ all the diagonal elements are zero,

$$\overline{F}^{\mu\mu} = F^{\mu\mu} = 0\,,\qquad \mu = 0,1,2,3$$

and we need to calculate only the elements $\overline{F}^{\mu\nu}$ with $\mu < \nu$:

$$\begin{aligned}\overline{F}^{12} &= \frac{1}{2}\varepsilon^{12\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{1230}F_{30} + \varepsilon^{1203}F_{03}\right) =\\ &= \frac{1}{2}\left(-F_{30} + F_{03}\right) = F^{30} = \frac{1}{c}E_z\,,\\ \overline{F}^{13} &= \frac{1}{2}\varepsilon^{13\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{1320}F_{20} + \varepsilon^{1302}F_{02}\right)\\ &= \frac{1}{2}\left(F_{20} - F_{02}\right) = F^{02} = -\frac{1}{c}E_y\,,\\ \overline{F}^{01} &= \frac{1}{2}\varepsilon^{01\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{0123}F_{23} + \varepsilon^{0132}F_{32}\right)\\ &= \frac{1}{2}\left(F_{23} - F_{32}\right) = F^{23} = -B_x\,,\\ \overline{F}^{23} &= \frac{1}{2}\varepsilon^{23\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{2310}F_{10} + \varepsilon^{2301}F_{01}\right)\\ &= \frac{1}{2}\left(-F_{10} + F_{01}\right) = F^{10} = \frac{1}{c}E_x\,,\end{aligned}$$

$$\begin{aligned}
\overline{F}^{02} &= \frac{1}{2}\varepsilon^{02\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{0213}F_{13} + \varepsilon^{0231}F_{31}\right) \\
&= \frac{1}{2}\left(-F_{13} + F_{31}\right) = F^{31} = -B_y \,, \\
\overline{F}^{03} &= \frac{1}{2}\varepsilon^{03\rho\sigma}F_{\rho\sigma} = \frac{1}{2}\left(\varepsilon^{0312}F_{12} + \varepsilon^{0321}F_{21}\right) \\
&= \frac{1}{2}\left(F_{12} - F_{21}\right) = F^{12} = -B_z \,.
\end{aligned}$$

The correctness of the statement is evident.

**Solution 2.5.11**

$$\Sigma \xrightarrow{v\mathbf{e}_z} \widehat{\Sigma} \,.$$

1. Point charge $q$ *'at rest'* at the origin of coordinates in $\widehat{\Sigma}$. Hence, the potentials in $\widehat{\Sigma}$ are:

$$\hat{\varphi} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\hat{r}} \,; \quad \hat{\mathbf{A}} = 0 \,.$$

Here

$$\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2 + \hat{z}^2} = \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2} \,.$$

On the right-hand side $\hat{r}$ is expressed by $\Sigma$-coordinates. Therewith the four-potential reads:

$$\hat{A}^\mu = \left(\frac{q}{4\pi\varepsilon_0 c}\frac{1}{\hat{r}}, 0, 0, 0\right) \,.$$

Potentials in $\Sigma$

$$A^\mu = \left(\hat{L}^{-1}\right)_{\mu\alpha}\hat{A}^\alpha \quad \left(\hat{L}^{-1} \equiv \hat{L}(v \to -v)\right) \,.$$

Transformation formulae:

$$\begin{aligned}
A^0 &= \gamma\hat{A}^0 + \gamma\beta\hat{A}^3 = \gamma\hat{A}^0 = \gamma\frac{q}{4\pi\varepsilon_0 c}\frac{1}{\hat{r}} \\
A^{1,2} &= \hat{A}^{1,2} = 0 \\
A^3 &= \gamma\beta\hat{A}^0 + \gamma\hat{A}^3 = \gamma\beta\hat{A}^0 = \gamma\beta\frac{q}{4\pi\varepsilon_0 c}\frac{1}{\hat{r}} \\
\curvearrowright\ A^\mu &= \frac{q}{4\pi\varepsilon_0}\frac{1}{\hat{r}}\gamma\frac{1}{c}\left(1, 0, 0, \beta\right) \,.
\end{aligned}$$

That means explicitly for the electromagnetic potentials:

$$\varphi(x,y,z,t) = \frac{q}{4\pi\varepsilon_0}\,\frac{\gamma}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}$$

$$\mathbf{A}(x,y,z,t) = \frac{q}{4\pi\varepsilon_0}\,\frac{v}{c^2}\,\frac{\gamma}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}\,(0,0,1)\ .$$

2. Fields in $\Sigma$:

   (a) Magnetic induction:

$$\mathbf{B} = \text{curl}\mathbf{A} = \left(\frac{\partial}{\partial y}A_z, -\frac{\partial}{\partial x}A_z, 0\right)\ .$$

   With

$$\frac{\partial}{\partial y}\frac{1}{\hat{r}} = -\frac{y}{\hat{r}^3}\ ;\quad \frac{\partial}{\partial x}\frac{1}{\hat{r}} = -\frac{x}{\hat{r}^3}$$

   it follows

$$\mathbf{B} = \frac{q}{4\pi\varepsilon_0}\,\gamma\,\frac{\beta}{c}\,\frac{1}{\hat{r}^3}\,(-y,x,0)\ .$$

   (b) Electric field:

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}} = -\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} + \dot{A}_z\right)\ .$$

   It is:

$$\nabla\varphi = \frac{q}{4\pi\varepsilon_0}\,\gamma\left(-\frac{1}{\hat{r}^3}\right)(x,y,\gamma^2(z-vt))$$

   Furthermore:

$$\dot{\mathbf{A}} = \frac{q}{4\pi\varepsilon_0}\,\gamma\,\frac{v}{c^2}\left(-\frac{1}{\hat{r}^3}\right)(0,0,-\gamma^2(z-vt)v)\ .$$

   This means eventually:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\,\gamma\,\frac{1}{\hat{r}^3}\,(x,y,z-vt)\ .$$

   Because of

$$\boldsymbol{\beta}\times\mathbf{E} = \beta(-E_y,E_x,0)$$

it follows obviously:

$$\mathbf{B} = \frac{1}{c}(\boldsymbol{\beta} \times \mathbf{E}) \ .$$

3. One recognizes:

$$\frac{1}{c^2}\,\frac{\partial^2 \varphi}{\partial t^2} = \frac{v^2}{c^2}\,\frac{\partial^2 \varphi}{\partial z^2}$$

and therewith

$$\begin{aligned}\Box\varphi &= \left(\triangle - \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\right)\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial z^2}\right)\varphi \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\gamma^2}\,\frac{\partial^2}{\partial z^2}\right)\varphi \ .\end{aligned}$$

We substitute:

$$u = z - vt\ ; \quad v = \frac{1}{\gamma}\,y \quad w = \frac{1}{\gamma}\,x \ .$$

This means:

$$\frac{\partial}{\partial x} = \frac{1}{\gamma}\,\frac{\partial}{\partial w}\ ; \quad \frac{\partial}{\partial y} = \frac{1}{\gamma}\,\frac{\partial}{\partial v}\ ; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial u} \ .$$

It follows:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\,\frac{\gamma}{\sqrt{\gamma^2 w^2 + \gamma^2 v^2 + \gamma^2 u^2}} = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{\sqrt{w^2 + v^2 + u^2}} \ .$$

It remains to be calculated:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\gamma^2}\,\frac{\partial^2}{\partial z^2}\right)\varphi = \frac{1}{\gamma^2}\left(\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial u^2}\right)\varphi \ .$$

On the other hand, one gets with $\mathbf{r} \equiv (w, v, u)$:

$$\Delta_{w,v,u}\,\frac{1}{r} = -4\pi\,\delta(\mathbf{r}) = -4\pi\,\delta(w)\delta(v)\delta(u) \ .$$

So one can write:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{\gamma^2}\frac{\partial^2}{\partial z^2}\right)\varphi = -\frac{q}{4\pi\varepsilon_0}4\pi\frac{1}{\gamma^2}\gamma\delta(x)\gamma\delta(y)\delta(z-vt)$$
$$= -\frac{q}{\varepsilon_0}\delta(x)\delta(y)\delta(z-vt) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}\ .$$

That is valid for the point charge:

$$\rho(x,y,z) = q\,\delta(x)\delta(y)\delta(z-vt)\ .$$

That was to be proven!

**Solution 2.5.12**

- In $\Sigma$:
  Lorentz force on the point charge $q$:

$$\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B}]\ .$$

The particle is at rest, i.e. $\mathbf{v}=0$. Thus:

$$\mathbf{F} = q\mathbf{E}\ .$$

- In $\Sigma'$:
  Velocity of the particle: $\mathbf{v}' = -\mathbf{v}_0$. Therewith the Lorentz force reads:

$$\mathbf{F}' = q\left(\mathbf{E}' + (-\mathbf{v}_0)\times\mathbf{B}'\right)\ .$$

- $\Sigma$, $\Sigma'$ are both inertial systems. Therefore it must hold: $\mathbf{F}=\mathbf{F}'$. This means here:

$$\mathbf{E}' = \mathbf{E}+\mathbf{v}_0\times\mathbf{B}' = \mathbf{E}+\alpha\mathbf{E}\times\mathbf{B}'\ .$$

Component in the direction of $\mathbf{E}$:

$$\frac{1}{E}\left(\mathbf{E}'\cdot\mathbf{E}\right) = \frac{1}{E}(\mathbf{E}\cdot\mathbf{E})+0 = E\ .$$

That was to be shown.

**Solution 2.5.13** Force on charge $q$ in $\Sigma$:

$$\mathbf{F} = q(\mathbf{E}+\mathbf{u}\times\mathbf{B})$$

with

$$\mathbf{u} = (a, b, d) \; ; \quad \mathbf{B} = (0, B, 0) \; ; \quad \mathbf{E} = \frac{1}{\sqrt{2}}(E, E, 0) \; .$$

Therefore:

$$\mathbf{u} \times \mathbf{B} = (-Bd, 0, Ba) \; .$$

That yields:

$$\mathbf{F} = q\left(\frac{1}{\sqrt{2}}E - Bd, \frac{1}{\sqrt{2}}E, Ba\right) \; .$$

Therewith we calculate

$$\mathbf{F} \cdot \mathbf{u} = q\left(\left(\frac{1}{\sqrt{2}}E - Bd\right)a + \frac{1}{\sqrt{2}}Eb + Bad\right) = \frac{1}{\sqrt{2}}qE(a+b) \; .$$

With the formulae (2.165)–(2.167) we then find the forces in $\Sigma'$:

$$F'_x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{q\left(\frac{1}{\sqrt{2}}E - Bd\right)}{1 - \frac{vd}{c^2}}$$

$$F'_y = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \frac{q\frac{1}{\sqrt{2}}E}{1 - \frac{vd}{c^2}}$$

$$F'_z = \frac{q\left(Ba - \frac{v}{c^2}\frac{1}{\sqrt{2}}E(a+b)\right)}{1 - \frac{vd}{c^2}} \; .$$

**Solution 2.5.14**

1. It holds according to (3.45), Vol. 3:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{r}\cdot\mathbf{m})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right) \; ,$$

$$B_x = \frac{\mu_0}{4\pi}\frac{3\,m\,z\,x}{r^5} \; ; \quad B_y = \frac{\mu_0}{4\pi}\frac{3\,m\,z\,y}{r^5} \; ,$$

$$B_z = \frac{\mu_0}{4\pi}\frac{m}{r^5}\left(2\,z^2 - x^2 - y^2\right) \; .$$

2. Let $\Sigma'$ be the rest system of the dipole. As calculated in part 1. we find there the fields:

$$
\begin{aligned}
B'_x &= \frac{\mu_0}{4\pi} \frac{3\, m\, z'\, x'}{r'^5} \; ; \quad B'_y = \frac{\mu_0}{4\pi} \frac{3\, m\, z'\, y'}{r'^5} \; , \\
B'_z &= \frac{\mu_0}{4\pi} \frac{m}{r'^5} \left( 2\, z'^2 - x'^2 - y'^2 \right) \; , \\
\mathbf{E}' &\equiv 0 \; ; \qquad r' = \sqrt{x'^2 + y'^2 + z'^2} \; .
\end{aligned}
$$

In $\Sigma$ we have according to (2.132) to (2.137):

$$
\begin{aligned}
B_x &= \gamma\, B'_x = \gamma \frac{\mu_0}{4\pi} 3m \frac{z' x'}{r'^5} \; , \\
&\qquad x' = x \; ; \;\; y' = y \; ; \;\; z' = \gamma(z - v\, t) \\
&\qquad \Longrightarrow \; r' = \sqrt{x^2 + y^2 + \gamma^2 (z - v\, t)^2} \\
\Longrightarrow \; B_x &= \gamma^2 \frac{\mu_0}{4\pi} 3m \frac{x(z - v\, t)}{\left[x^2 + y^2 + \gamma^2 (z - v\, t)^2\right]^{5/2}} \; , \\
B_y &= \gamma\, B'_y = \gamma^2 \frac{\mu_0}{4\pi} 3m \frac{y(z - v\, t)}{\left[x^2 + y^2 + \gamma^2 (z - v\, t)^2\right]^{5/2}} \; , \\
B_z &= B'_z = \frac{\mu_0}{4\pi} m \frac{2\gamma^2 (z - v\, t)^2 - x^2 - y^2}{\left[x^2 + y^2 + \gamma^2 (z - v\, t)^2\right]^{5/2}} \; , \\
E_x &= +\gamma\, \beta\, c\, B'_y = v\, B_y \; , \\
E_y &= -\gamma\, \beta\, c\, B'_x = -v\, B_x \; , \\
E_z &= E'_z = 0 \; .
\end{aligned}
$$

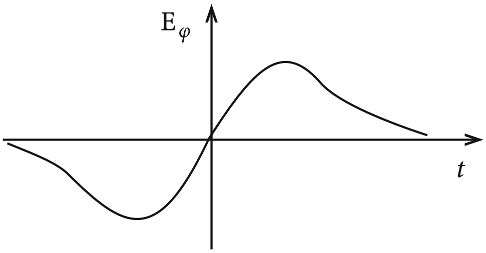
3.

$$
\mathbf{e}_\rho = (\cos\varphi, \sin\varphi, 0) \; ; \quad \mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \; ; \quad \mathbf{e}_z = (0, 0, 1) \; .
$$

Electric field:

$$
\begin{aligned}
E_\rho &= \mathbf{E} \cdot \mathbf{e}_\rho = E_x \cos\varphi + E_y \sin\varphi = v \left( B_y \cos\varphi - B_x \sin\varphi \right) \\
&= \ldots\, (y \cos\varphi - x \sin\varphi) = \ldots\, (\rho \sin\varphi \cos\varphi - \rho \cos\varphi \sin\varphi) = 0 \; , \\
E_\varphi &= \mathbf{E} \cdot \mathbf{e}_\varphi = -E_x \sin\varphi + E_y \cos\varphi = -v \left( B_y \sin\varphi + B_x \cos\varphi \right) \\
&= \gamma^2 \frac{\mu_0}{4\pi} 3m \frac{(z - v\, t)}{\left(x^2 + y^2 + \gamma^2 (z - v\, t)^2\right)^{5/2}} \left( -v\, y \sin\varphi - v\, x \cos\varphi \right)
\end{aligned}
$$

Fig. A.3




$$= \ldots - v\,\rho\left(\sin^2\varphi + \cos^2\varphi\right) = \gamma^2\frac{\mu_0}{4\pi}\,3m\,\frac{-v(z - v\,t)\rho}{\left[\rho^2 + \gamma^2(z - v\,t)^2\right]^{5/2}}\ ,$$

$$E_z = 0\ ,$$

$$B_\rho = B_x\cos\varphi + B_y\sin\varphi = \frac{1}{v}\left(E_x\sin\varphi - E_y\cos\varphi\right) = -\frac{1}{v}E_\varphi\ ,$$

$$B_\varphi = -B_x\sin\varphi + B_y\cos\varphi = \frac{1}{v}\left(E_y\sin\varphi + E_x\cos\varphi\right) = \frac{1}{v}E_\rho = 0\ ,$$

$$B_z = \frac{\mu_0}{4\pi}\,m\,\frac{2\gamma^2(z - v\,t)^2 - \rho^2}{\left[\rho^2 + \gamma^2(z - v\,t)^2\right]^{5/2}}\ .$$

4. The **E**-lines are circles in the $xy$-plane with their centers at the origin of coordinates (Fig. A.3):

$$\mathbf{E} = E_\varphi(t)\,\mathbf{e}_\varphi\ ,$$

$$E_\varphi(t;\ z = 0) = \gamma^2\frac{\mu_0}{4\pi}\,3m\,\frac{v^2 t}{(\rho^2 + \gamma^2 v^2 t^2)^{5/2}}\ .$$

5. Closed **E**-lines can appear, because of curl$\mathbf{E} = -\dot{\mathbf{B}}$, when a time-dependent magnetic field is present.

## Solution 2.5.15

1. A charge, which is at first at rest in $\Sigma$, experiences in $\Sigma'$ an acceleration due to the pure $\mathbf{B}'$-field since the charge is moving in this system. In $\Sigma$, however, this acceleration must stem from an **E**-field, since the charge is initially at rest.
2. Let the $\mathbf{E}'$-field in $\Sigma'$ be produced by charges resting in $\Sigma'$, e.g. by charges on the plates of a capacitor. In $\Sigma$ these charges are moving thereby generating a **B**-field.

**Solution 2.5.16**

1. For the space components of the Minkowski force it holds:

$$\begin{aligned}\frac{d}{d\tau}\mathbf{p}_\mathrm{r} &= \gamma\frac{d}{dt}\mathbf{p}_\mathrm{r} = \gamma\,\mathbf{F} = \gamma\,q(\mathbf{v}\times\mathbf{B}),\\ \mathbf{p}_\mathrm{r} &= \gamma\,m\,\mathbf{v}\\ \Longrightarrow\ \mathbf{p}_\mathrm{r}\cdot\frac{d}{dt}\mathbf{p}_\mathrm{r} &= \gamma\,m\,q\ \mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0 = \frac{1}{2}\frac{d}{dt}\mathbf{p}_\mathrm{r}^2\end{aligned}$$

$$\Longrightarrow\ \mathbf{p}_\mathrm{r}^2 = \text{const}\ \Longrightarrow\ v^2 = \text{const}\ \Longleftrightarrow\ \gamma(v) = \left(1-\frac{v^2}{c^2}\right)^{-1/2} = \text{const}\ .$$

The relativistic energy follows (2.63):

$$T_\mathrm{r} = \sqrt{c^2\mathbf{p}_\mathrm{r}^2 + m^2c^4} = \text{const}\ .$$

2. Initial condition: $p_\mathrm{r}(t=0) = m\,\gamma\,v_0(1,0,0)$

$$\begin{aligned}\dot{\mathbf{p}}_\mathrm{r} = q(\mathbf{v}\times\mathbf{B}) &= q\left(v_y\,B, -v_x\,B, 0\right)\\ &= \frac{1}{\gamma}\frac{q\,B}{m}\left(p_{\mathrm{r}y}, -p_{\mathrm{r}x}, 0\right)\ .\end{aligned}$$

First partial result: $p_{\mathrm{r}z} = \text{const}$

$$\omega \equiv \frac{q\,B}{m}\frac{1}{\gamma} \equiv \omega_0\frac{1}{\gamma} \qquad (\gamma = \text{const})\ .$$

$$\begin{aligned}\frac{d}{dt}\left(p_{\mathrm{r}x} + i\,p_{\mathrm{r}y}\right) &= \omega\left(p_{\mathrm{r}y} - i\,p_{\mathrm{r}x}\right) = -i\,\omega\left(p_{\mathrm{r}x} + i\,p_{\mathrm{r}y}\right)\\ \Longrightarrow &\quad \left(p_{\mathrm{r}x} + i\,p_{\mathrm{r}y}\right)(t) = (p_{\mathrm{r}x} + i\,p_{\mathrm{r}y})(0)\,e^{-i\omega t}\\ \Longrightarrow &\quad p_{\mathrm{r}x}(t) = p_{\mathrm{r}x}(0)\cos\omega t + p_{\mathrm{r}y}(0)\sin\omega t\ ,\\ &\quad p_{\mathrm{r}y}(t) = -p_{\mathrm{r}x}(0)\sin\omega t + p_{\mathrm{r}y}(0)\cos\omega t\ .\end{aligned}$$

Initial conditions:

$$\mathbf{p}_\mathrm{r}(0) = m\,\gamma\,(v_0, 0, 0)\ .$$

This yields:

$$\mathbf{p}_\mathrm{r}(t) = m\,\gamma\,v_0(\cos\omega t, -\sin\omega t, 0)\ .$$

$\mathbf{p}_\mathrm{r}^2(t) = \text{const}$ is obviously guaranteed.

3. The trajectory we get from

$$\mathbf{r}(t) - \mathbf{r}(t=0) = \frac{1}{m\gamma}\int_0^t dt'\,\mathbf{p}_\mathrm{r}(t')$$
$$= \frac{v_0}{\omega}\left(\sin\omega t', \cos\omega t', 0\right)\Big|_0^t$$
$$\Longrightarrow\ \mathbf{r}(t) = (0, y_0, 0) + \gamma\frac{m\,v_0}{q\,B}\{(\sin\omega t, \cos\omega t, 0) - (0,1,0)\}$$
$$\Longrightarrow\ \mathbf{r}(t) = \gamma\frac{m\,v_0}{q\,B}(\sin\omega t, \cos\omega t, 0)\ .$$

**Solution 2.5.17**

1. Space components of the relativistic momentum:

$$\mathbf{p}_\mathrm{r} = \gamma(v)m\,\mathbf{v}\ ; \qquad \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\ ,$$
$$\dot{\mathbf{p}}_\mathrm{r} = q\,\mathbf{E}$$
$$\Longrightarrow\ \dot{p}_{\mathrm{r}x} = q\,E\ ; \quad \dot{p}_{\mathrm{r}y} = \dot{p}_{\mathrm{r}z} = 0\ .$$

After integration it follows with the given initial conditions:

$$\mathbf{p}_\mathrm{r}(t) = (q\,E\,t,\ \gamma_0\,m\,v_0, 0)\ ; \quad \gamma_0 = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}\ .$$

Using (2.63) we get then for the relativistic kinetic energy:

$$T_\mathrm{r}(t) = \sqrt{m^2c^4 + c^2p_\mathrm{r}^2(t)} = \sqrt{m^2c^4 + c^2(q^2E^2t^2 + \gamma_0^2m^2v_0^2)}\ .$$

2. Notice, differently from the case of a homogeneous magnetic field (Exercise 2.5.16), in the homogeneous electric field $v^2 = v^2(t)$. $\gamma(v)$ is thus not a constant, but it is according to (2.61):

$$T_\mathrm{r} = m\,\gamma(v)c^2$$
$$\Longrightarrow\ \mathbf{p}_\mathrm{r} = \frac{T_\mathrm{r}}{c^2}\mathbf{v}\ \Longleftrightarrow\ \mathbf{v} = \frac{c^2}{T_\mathrm{r}}\mathbf{p}_\mathrm{r}\ .$$

In detail this means:

$$\dot{x}(t) = \frac{c^2\, q\, E\, t}{\sqrt{m^2c^4 + c^2\left(q^2E^2t^2 + \gamma_0^2 m^2 v_0^2\right)}} = \frac{1}{q\,E}\,\frac{d}{dt}\,T_{\rm r}(t)\ ,$$

$$\begin{aligned}
\dot{y}(t) &= \frac{c^2\gamma_0 m\, v_0}{\sqrt{m^2c^4 + c^2\left(q^2E^2t^2 + \gamma_0^2 m^2 v_0^2\right)}} \\
&= \frac{c^2\gamma_0 m\, v_0}{\sqrt{c^2q^2E^2t^2 + T_{\rm r}^2(0)}} \\
&= \frac{c^2\gamma_0 m\, v_0}{T_{\rm r}(0)}\,\frac{1}{\sqrt{\left(\frac{c\,q\,E\,t}{T_{\rm r}(0)}\right)^2 + 1}} \\
&= \frac{c^2\gamma_0 m\, v_0}{T_{\rm r}(0)}\left(\frac{d}{dx}\operatorname{arcsinh} x\right)_{x=\frac{c\,q\,E\,t}{T_{\rm r}(0)}} \\
&= \frac{c^2\gamma_0 m\, v_0}{T_{\rm r}(0)}\,\frac{T_{\rm r}(0)}{cqE}\,\frac{d}{dt}\operatorname{arcsinh}\left(\frac{c\,q\,E\,t}{T_{\rm r}(0)}\right) \\
&= \frac{c\gamma_0 m\, v_0}{qE}\,\frac{d}{dt}\operatorname{arcsinh}\left(\frac{c\,q\,E\,t}{T_{\rm r}(0)}\right)
\end{aligned}$$

$$\dot{z}(t) = 0\ .$$

3. Using the initial conditions we get:

$$\begin{aligned}
z(t) &\equiv z_0\ , \\
y(t) &= \frac{c\,\gamma_0 m\, v_0}{q\,E}\operatorname{arcsinh}\left[\frac{c\,q\,E\,t}{T_{\rm r}(0)}\right]\ , \\
x(t) &= \frac{1}{q\,E}\left(T_{\rm r}(t) - T_{\rm r}(0)\right)\ , \\
T_{\rm r}(0) &= \sqrt{m^2c^4 + c^2\gamma_0^2 m^2 v_0^2}\ .
\end{aligned}$$

The particle traverses the space curve $x = x(y)$:

$$c\,q\,E\,t = T_{\rm r}(0)\sinh\left(\frac{y\,q\,E}{c\,\gamma_0 m\, v_0}\right)\ ,$$

$$\begin{aligned}
T_{\rm r}(t) &= \sqrt{T_{\rm r}^2(0) + c^2q^2E^2t^2} = T_{\rm r}(0)\sqrt{1 + \sinh^2\left(\frac{q\,E}{c\,\gamma_0 m\, v_0}y\right)} \\
&= T_{\rm r}(0)\cosh\left(\frac{q\,E}{c\,\gamma_0 m\, v_0}y\right)\ .
\end{aligned}$$

From that it follows:

$$x = \frac{T_r(0)}{q\,E}\left[\cosh\left(\frac{q\,E}{c\,\gamma_0 m\,v_0}y\right) - 1\right] = x(y)\ .$$

**Solution 2.5.18**

1. It holds according to (2.59) and (2.54):

$$\frac{m(v)}{m(0)} = \frac{T_r(v)}{T_r(0)} = \gamma(v) = \frac{0.711}{0.511} = 1.391$$
$$\Longrightarrow\ m(v) = 1.391\,m(0)\ .$$

2.

$$\gamma = \left(1-\beta^2\right)^{-(1/2)} \iff \beta = \sqrt{\frac{\gamma^2-1}{\gamma^2}} = 0.695$$
$$\Longrightarrow\ v = 0.695\,c\ .$$

3.

$$v_{nr}^2 = \frac{2\,T}{m(0)} \Longrightarrow \frac{v_{nr}^2}{c^2} = \frac{2\,T}{m\,c^2} = \frac{0.4}{0.511}$$
$$\Longrightarrow\ v_{nr} = 0.885\,c\ .$$

Relative error:

$$\varepsilon = \frac{v_{nr} - v}{v}\,100 = 27.30\,\%\ .$$

**Solution 2.5.19**

$$m:\ \ mass\ :\quad \text{Lorentz invariant}\ ,$$
$$E_0 = mc^2\ :\quad \text{rest energy}\ ,$$
$$T_r = \frac{mc^2}{\sqrt{1 - v^2/c^2}}\ :\quad \text{kinetic energy}\ ,$$
$$T_r = \sqrt{c^2\mathbf{p}_r^2 + m^2c^4}\ .$$

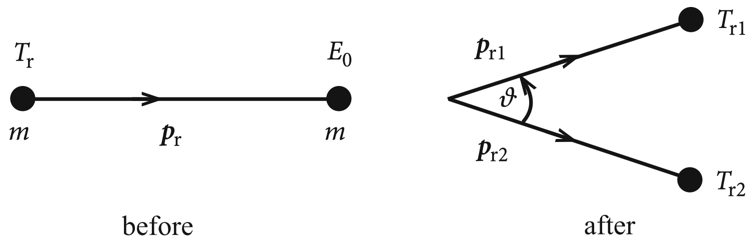



**Fig. A.4**

Elastic collision between two equal masses (Fig. A.4):

Goal: Calculation of the scattering angle $\vartheta$ as a function of $T_r$ and $T_{r1}$ (non-relativistic: $\vartheta = 90°$).

Law of conservation of momentum:

$$\mathbf{p}_r + \mathbf{0} = \mathbf{p}_{r1} + \mathbf{p}_{r2} \ .$$

Law of energy:

$$\begin{aligned}
T_r + E_0 &= T_{r1} + T_{r2} \ , \\
T_r^2 &= c^2 p_r^2 + E_0^2 \quad \Longrightarrow \quad p_{r_{1,2}} = \frac{1}{c}\sqrt{T_{r_{1,2}}^2 - E_0^2} \ , \\
p_r^2 &= p_{r1}^2 + p_{r2}^2 + 2\,p_{r1}p_{r2}\cos\vartheta \ , \\
\frac{1}{c^2}\left(T_r^2 - E_0^2\right) &= \frac{1}{c^2}\left(T_{r1}^2 + T_{r2}^2\right) - 2\frac{E_0^2}{c^2} + 2\,p_{r1}p_{r2}\cos\vartheta \\
\Longrightarrow \quad \cos\vartheta &= \frac{T_r^2 + E_0^2 - T_{r1}^2 - T_{r2}^2}{2\,p_{r1}p_{r2}\,c^2} \ .
\end{aligned}$$

Let us eliminate $T_{r2}, p_{r2}$ by conservation laws:

$$\begin{aligned}
T_{r2} &= T_r + E_0 - T_{r1} \ , \\
T_{r2}^2 &= T_r^2 + E_0^2 + T_{r1}^2 + 2\,T_r E_0 - 2\,T_r T_{r1} - 2\,E_0 T_{r1} \\
\Longrightarrow \quad T_r^2 + E_0^2 - T_{r1}^2 - T_{r2}^2 &= -2\,T_{r1}^2 - 2\,T_r E_0 + 2\,T_r T_{r1} + 2\,E_0 T_{r1} \\
&= 2\,(T_r - T_{r1})\,(T_{r1} - E_0) \ . \\
p_{r2} &= \frac{1}{c}\sqrt{T_{r2}^2 - E_0^2} = \frac{1}{c}\sqrt{(T_{r2} + E_0)\,(T_{r2} - E_0)} \\
&= \frac{1}{c}\sqrt{(T_r - T_{r1})\,(T_r + 2E_0 - T_{r1})} \\
p_{r1} &= \frac{1}{c}\sqrt{(T_{r1} + E_0)\,(T_{r1} - E_0)}
\end{aligned}$$

$$\Longrightarrow \quad \cos\vartheta = \frac{(T_r - T_{r1})\,(T_{r1} - E_0)}{\sqrt{(T_{r1} + E_0)\,(T_{r1} - E_0)\,(T_r - T_{r1})\,(T_r + 2\,E_0 - T_{r1})}}\ ,$$

$$\cos\vartheta = \frac{1}{\sqrt{\left(1 + \frac{2\,E_0}{T_{r1} - E_0}\right)\left(1 + \frac{2\,E_0}{T_r - T_{r1}}\right)}}\ .$$

$$T_{r1,2} \geq E_0\ : \qquad T_r \geq T_{r1} \quad \Longrightarrow \quad \cos\vartheta < 1 \Longrightarrow \vartheta_{\min} < \vartheta \leq 90^\circ\ .$$

(a)

$$v \approx c \quad \Longrightarrow \quad T_r, T_{r_1}, T_{r_2} \gg E_0\ ,$$

$$T_r - T_{r_1} = T_{r_2} - E_0 \gg E_0$$

$$\cos\vartheta_{\min} \to 1\ ; \quad \vartheta_{\min} \to 0^\circ\ .$$

(b)

$$v \ll c \quad \Longrightarrow \quad \frac{2\,E_0}{T_{r1} - E_0} \gg 1\ ; \quad \frac{2\,E_0}{T_r - T_{r1}} \gg 1$$

$$\Longrightarrow \quad \cos\vartheta \to 0\ ; \quad \vartheta \to 90^\circ\ .$$

It thus comes out the known non-relativistic result!

**Solution 2.5.20**

1.

$$T_r = T + T_r(0) = 2\,T_r(0) \stackrel{(2.54)}{=} \gamma\, T_r(0) \ \Longrightarrow\ \gamma = 2\ .$$

This means:

$$\beta = \left(\frac{\gamma^2 - 1}{\gamma^2}\right)^{1/2} = \sqrt{\frac{3}{4}} = 0.866\ .$$

The velocity of the $\pi^+$-meson therefore amounts to:

$$v\left(\pi^+\right) = 0.866\,c = 2.598 \cdot 10^8\ \frac{\text{m}}{\text{s}}\ .$$

2. Decay time in the rest system of the meson:

$$\tau = 2.5 \cdot 10^{-8}\,\text{s}\ .$$

Decay time in the rest system of the observer:

$$\tau' = \gamma\,\tau = 5\cdot 10^{-8}\,\mathrm{s}\ .$$

*'Decay-distance'*:

$$d = v\,\tau' = 12.990\,\mathrm{m}\ .$$

**Solution 2.5.21** Law of conservation of energy:

$$T_{\mathrm{r}}(v) + T_{\mathrm{r}}(0) = T_{\mathrm{r}}(v_1) + T_{\mathrm{r}}(v_2)$$
$$\Longrightarrow\ \gamma(v) + 1 = \gamma(v_1) + \gamma(v_2)\ .$$

Law of conservation of momentum:

$$m\,\gamma\,\mathbf{v} = m\,\gamma(v_1)\,\mathbf{v}_1 + m\,\gamma(v_2)\,\mathbf{v}_2$$
$$\Longrightarrow\ \gamma(v)\boldsymbol{\beta} = \gamma(v_1)\,\boldsymbol{\beta}_1 + \gamma(v_2)\,\boldsymbol{\beta}_2\ .$$

The last equation may be decomposed into components:

$$\gamma(v)\beta = \gamma(v_1)\,\beta_1\cos\vartheta + \gamma(v_2)\,\beta_2\cos\varphi\ ,$$
$$0 = \gamma(v_1)\,\beta_1\sin\vartheta - \gamma(v_2)\,\beta_2\sin\varphi\ ,$$
$$\gamma^2(v_2)\,\beta_2^2\left(\cos^2\varphi + \sin^2\varphi\right) = \gamma^2(v_1)\,\beta_1^2\sin^2\vartheta + \left(\gamma(v)\beta - \gamma(v_1)\,\beta_1\cos\vartheta\right)^2$$
$$\Longrightarrow\ \gamma^2(v_2)\,\beta_2^2 = \gamma^2(v_1)\,\beta_1^2 + \gamma^2(v)\beta^2 - 2\gamma(v)\gamma(v_1)\,\beta\beta_1\cos\vartheta\ .$$

We eliminate the $\boldsymbol{\beta}$'s by the relation:

$$\gamma(v_i)\,\beta_i = \sqrt{\gamma^2(v_i) - 1}$$
$$\Longrightarrow\ \gamma^2(v_2) - 1 = \gamma^2(v_1) - 1 + \gamma^2(v) - 1 - 2\sqrt{\gamma^2(v) - 1}\,\sqrt{\gamma^2(v_1) - 1}\cos\vartheta\ .$$

We combine this equation with the law of conservation of energy in order to remove $\gamma(v_2)$:

$$\gamma(v_2) = \gamma(v) + 1 - \gamma(v_1)$$
$$\Longrightarrow\ \gamma^2(v_2) = \gamma^2(v) + 1 + \gamma^2(v_1) + 2\gamma(v) - 2\gamma(v_1) - 2\gamma(v)\gamma(v_1)\ .$$

This we insert into the above equation:

$$\begin{aligned} 2\gamma(v) - 2\gamma\,(v_1) - 2\gamma(v)\gamma\,(v_1) &= -2 - 2\sqrt{\gamma^2(v) - 1}\sqrt{\gamma^2\,(v_1) - 1}\cos\vartheta \\ \Longrightarrow\;\; -2\,(\gamma\,(v_1) - 1)\;(\gamma(v) + 1) &= -2\sqrt{\gamma^2(v) - 1}\;\sqrt{\gamma^2\,(v_1) - 1}\cos\vartheta \\ \Longrightarrow\;\; (\gamma\,(v_1) - 1)\;(\gamma(v) + 1) &= (\gamma(v) - 1)\;(\gamma\,(v_1) + 1)\cos^2\vartheta\;. \end{aligned}$$

That can be solved for $\gamma(v_1)$:

$$\gamma\,(v_1) = \frac{(\gamma(v) + 1) + (\gamma(v) - 1)\cos^2\vartheta}{(\gamma(v) + 1) - (\gamma(v) - 1)\cos^2\vartheta}\;.$$

With the abbreviation

$$\alpha^2 = \frac{\gamma(v) - 1}{\gamma(v) + 1}$$

it then remains to be solved:

$$\gamma\,(v_1) = \frac{1 + \alpha^2\,\cos^2\vartheta}{1 - \alpha^2\,\cos^2\vartheta}\;.$$

We now go back to the law of conservation of momentum and build the quotient of the two component equations:

$$\tan\varphi = \frac{\gamma\,(v_1)\,\beta_1\sin\vartheta}{\gamma(v)\beta - \gamma\,(v_1)\,\beta_1\cos\vartheta}\;.$$

It is

$$\begin{aligned} \gamma\,(v_1)\,\beta_1 &= \sqrt{\gamma^2\,(v_1) - 1} = \sqrt{\frac{\left(1 + \alpha^2\,\cos^2\vartheta\right)^2}{\left(1 - \alpha^2\cos^2\vartheta\right)^2} - 1} \\ &= \frac{2\alpha\cos\vartheta}{1 - \alpha^2\cos^2\vartheta}\;. \end{aligned}$$

On the other hand it holds:

$$\gamma(v)\beta = \sqrt{\gamma^2(v) - 1} = \sqrt{(\gamma(v) + 1)\,(\gamma(v) - 1)} = \alpha\,(\gamma(v) + 1)\;\;.$$

This we insert into the expression for $\tan\varphi$:

$$\begin{aligned}\tan\varphi &= \frac{\frac{2\alpha\cos\vartheta\sin\vartheta}{1-\alpha^2\cos^2\vartheta}}{\alpha\,(\gamma(v)+1)-\frac{2\alpha\cos^2\vartheta}{1-\alpha^2\cos^2\vartheta}}\\ &= \frac{2\cos\vartheta\sin\vartheta}{(\gamma(v)+1)-(\gamma(v)+1)\,\alpha^2\cos^2\vartheta-2\cos^2\vartheta}\\ &= \frac{2\cos\vartheta\sin\vartheta}{\gamma(v)\,(1-\cos^2\vartheta)+(1-\cos^2\vartheta)}\\ &= \frac{2}{\gamma(v)+1}\,\frac{\cos\vartheta\sin\vartheta}{1-\cos^2\vartheta}\ .\end{aligned}$$

From this it follows eventually the assertion:

$$\tan\varphi\,\tan\vartheta = \frac{2}{\gamma(v)+1}\ .$$

In the non-relativistic limit $\gamma\to 1$ we get:

$$\tan\varphi\,\tan\vartheta \to 1\ .$$

Because of

$$\tan(\varphi+\vartheta) = \frac{\tan\vartheta+\tan\varphi}{1-\tan\varphi\tan\vartheta}$$

this means

$$\tan(\varphi+\vartheta)\to\infty$$

and therewith:

$$\varphi+\vartheta\to\frac{\pi}{2}\ .$$

**Solution 2.5.22** Energy of the electron:

$$T_\mathrm{r}^2 = c^2p_\mathrm{r}^2+m^2c^4 = c^2p_\mathrm{r}^2+T_\mathrm{r}^2(0)\ .$$

On the other hand:

$$T_\mathrm{r} = T+T_\mathrm{r}(0)\ .$$

Combination of these two equations:

$$\begin{aligned} T^2 + 2\,T\,T_{\mathrm{r}}(0) &= c^2 p_{\mathrm{r}}^2 \;, \\ T &\longleftrightarrow 1\,\mathrm{MeV} \;, \\ T_{\mathrm{r}}(0) &\longleftrightarrow 0.511\,\mathrm{MeV} \\ \Longrightarrow\;\; c^2 p_{\mathrm{r}}^2 = 2.022\,(\mathrm{MeV})^2 &\Longrightarrow\;\; c\,p_{\mathrm{r}} = 1.422\,\mathrm{MeV} \;. \end{aligned}$$

The photon has no rest mass:

$$T_{\mathrm{r}}(\gamma) = c\,p_{\mathrm{r}} = 1.422\,\mathrm{MeV} \;.$$

# Index





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